p-adic formulas and unit root F-subcrystals of the hypergeometric system

by

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0. Introduction.

This article is dedicated to the study of p-adic analytic continuation of the unit root F-subcrystal of a logarithmic F-crystal in the open tube of a singularity. We will show how the possibility of extending that unit root crystal as a *non-singular* crystal in more singular classes, originates non-trivial formulas of analytic continuation of classical functions. As Katz puts it in [Ka], this should be considered a formula of analytic continuation *par excellence*. We improve on Katz' treatment at least in that we allow logarithmic singularities. A typical example of a formula of this type is the Koblitz-Diamond formula [Ko], [D], for the analytic continuation of the function $\mathcal{F}(a,b,c;\lambda)$ [p-DE IV] related to the classical Gauss hypergeometric function $F(a,b,c;\lambda)$, for a, b, $c\in\mathbb{Z}_p$, $c\notin\mathbb{Z}_{\leqslant 0}$. We recall that, for any $(a,b,c)\in(\mathbb{Z}_p)^3$ for which it makes sense,

$$F(a,b,c;\lambda)=\sum_{s=0}^{+\infty}\frac{(a)_s(b)_s}{(c)_s s!}\lambda^s\in\mathbb{Q}_p[[\lambda]]\ ,$$

where, as usual, we use Pochammer's notation $(a)_s=a(a+1)\dots(a+s-1)$. The function $\mathcal{F}(a,b,c;\lambda)$ is the maximal p-adic analytic extension of the ratio

$$\frac{F(a,b,c;\lambda)}{F(a',b',c';\lambda^p)}\in 1+\lambda\mathbb{Q}_p\,[\![\lambda]\!]$$

where for $a\in\mathbb{Z}_p$, $a'\in\mathbb{Z}_p$ is uniquely defined by the condition that $pa'-a=\mu_a\in\{0,1,\dots,p-1\}$ (We also recursively define $a^{(0)}=a$, and $a^{(i+1)}=(a^{(i)})'$, for $i=0,1,\dots$). The Koblitz-Diamond formula asserts that if $c^{(i)}\in\mathbb{Z}_p^\times$ and $\mu_{c^{(i)}}\geqslant\mu_{a^{(i)}}+\mu_{b^{(i)}}$ for any $i=0,1,\dots$, then $\mathcal{F}(a,b,c;\lambda)$ extends analytically to the open disk of radius 1 around $\lambda=1$ and

$$\mathcal{F}(a,b,c;1)=\frac{\Gamma_p(c)\Gamma_p(c-a-b)}{\Gamma_p(c-a)\Gamma_p(c-b)}$$

(where Γ_p denotes the Morita p-adic gamma function), see [D], and [Ko] for $c=1$. A similar discussion, in the non-singular case, appears in [Ba], where it is used to explain a formula of Young [Yo]: If $\mu_{a^{(i)}}\leqslant\mu_{b^{(i)}}<p-1$, $\mu_{a^{(i)}}$ even, and $2\mu_{b^{(i)}}-\mu_{a^{(i)}}\leqslant p-1$ for all $i\in\mathbb{N}$ then $\mathcal{F}(a,b,1+a-b;\lambda)$ admits an analytic extension to the class of -1, and

$$\mathcal{F}(a,b,1+a-b;-1)=(-)^{\frac{\mu_a}{2}}\frac{\Gamma_p\left(\frac{a}{2}\right)\Gamma_p\left(b-\frac{a}{2}\right)}{\Gamma_p(a)\Gamma_p(b-a)}\ .$$

The original paper by Young needs the further condition that a,b be rational (so in $\frac{1}{p^f-1}\mathbb{Z}$, for some $f\in\mathbb{Z}_{\geqslant 1}$). The generalization and the interpretation in terms of unit-root F-subcrystal is given in [Ba].

The reader may have noticed already that our discussion goes beyond the classical theory [Ka] of F-crystals, even if extended to the framework of logarithmic schemes [Sh]. What we are really dealing with is a *Dwork family of (filtered, logarithmic) F-crystals*: a structure which was

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introduced by the first author in lectures at a p-adic Summer School held in Trento in June 1995, inspired on Dwork's treatment of hypergeometric differential modules [GHF]. Since those lectures have unfortunately remained, as of today, unpublished, we quickly define Dwork families of F-crystals in section 2 below, with the promise of making available a more satisfactory treatment as soon as possible. The main point is that our (logarithmic) crystals $M(a,b,c)$ on, say, the formal p-adic base $\hat{X} = \hat{\mathbb{P}} := p$-adic completion of $\mathbb{P}^1_{\mathbb{Z}_p}$, with the log-structure induced by the three $\mathbb{Z}_p$-points $\{0,1,\infty\}$ (or $\hat{X} = \mathrm{Spf}\ \mathbb{Z}_p\{\lambda, \frac{1}{\lambda(1-\lambda)}\}$, if one prefers to avoid singularities) depend on parameters a, b, c in $\mathbb{Z}_p$, and Frobenius is a horizontal transformation $F(\varphi) : \varphi^* M(a',b',c') \longrightarrow M(a,b,c)$, where φ is some lifting of the absolute Frobenius to (some open subset of) $\hat{X}$. This is reminiscent of Mazur's theory of F-spans [Mz], but differs from it in one crucial point. The underlying $\mathcal{O}_{\hat{X}}$-module of $M(a,b,c)$ is here independent of a,b,c, while the connection $\nabla_{a,b,c}$ varies with the parameters. In that respect, the hypergeometric $M(a,b,c)$ is simply $\mathcal{O}^2_{\hat{\mathbb{P}}}$, and we will always express the connection and the Frobenius map in terms of this trivialization. This choice of global basis for the hypergeometric module, is also compatible with the filtration, which however varies with (a,b,c) (morally, with (a,b,c) mod p). When a,b,c are rational numbers, say in $(p^f-1)^{-1}\mathbb{Z}$, after f steps Frobenius gets us back to $M(a,b,c)$, and an f-th iterate of Frobenius becomes the standard semilinear automorphism of classical F-isocrystals, with respect to φ^f.

The notion of a Dwork family of F-crystals is actually taken to be more flexible than what we just said. According to Dwork's taste, we want to allow for integral translations of the parameters a, b, c, to reflect the existence of Gauss' contiguity relations on classical hypergeometric functions [Po], [GHF], [Bo], [Ku], [Ba]. These relations can be used in great generality to determine the finite-difference equation behavior of the Frobenius matrix, and they determine its shape, at least under the assumption that that matrix be p-adic meromorphic as a function of the parameters a, b, c in $\mathbb{Z}_p$. This assumption is known as the *Boyarsky principle*, and is know to hold for hypergeometric functions [GHF, 4.7.1]. The matrix $\gamma^{(\varphi)}(\vec{a}, \vec{b}; \lambda)$ used in this article is the one of [Bo, 3.1] and [Ku, 1.12.3]. (When $\vec{a} = (a,b,c) \in \mathbb{Z}_p^3$, $\vec{b} = (a',b',c')$ and $\varphi(\lambda) = \lambda^p$, it coincides with the matrix $\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$ of [LDE, 4.5.1]). The precise relation between the Frobenius matrix described in [GHF, 4.7.1] and the one of [LDE, 4.5.1] is given in [Ba, 2.21]. We use this finite-difference method in the appendix to provide an alternative calculation of the dominant polar term of the Frobenius matrix in the singular classes at 0 and 1.

The main assumption in this paper is the splitting of the Frobenius matrix of the hypergeometric family in the following cases [LDE, 6.6]:

(1) $\mu_c < \min(\mu_a, \mu_b)$,

(2) $\mu_c > \max(\mu_a, \mu_b)$.

If a,b,c are rational, this type of condition leads to the existence of a unit root F-subcrystal (of rank one) of the logarithmic F-crystal of rank two associated to the hypergeometric system.

The p-adic theory of the hypergeometric system

$$\frac{d}{d\lambda} Y = Y \begin{pmatrix} -\frac{c}{\lambda} & \frac{c-a}{1-\lambda} \\ \frac{c-b}{\lambda} & \frac{a+b-c}{1-\lambda} \end{pmatrix}$$

has been deeply investigated by Dwork [LDE]. The explanation of the Koblitz-Diamond formula preliminarly requires the calculation of the eigenvalues of Frobenius operating on the eigenvectors of classical monodromy in the class of a single logarithmic singularity. For this calculation to make sense, we must ensure the convergence of the uniform part of the classical fundamental solution matrix at the singular points in an open disk of radius 1. This forces us into a rather bizarre domain for $(a,b,c) \in \mathbb{Z}_p^3$, ensuring that all exponents and exponent differences of our differential system consist of p-adically non-Liouville numbers. The standard transfer theorems of p-adic analysis can then be applied [Ch], [BC2], [DGS, Chap. 6]. For the hypergeometric system, the singular class of 0, and with further restrictions on (a,b,c) (*e.g.* that they be in $(\mathbb{Z}_p \cap \mathbb{Q})^3$), this is the difficult computation of chapters 24 to 26 of [LDE].

To transfer that information to the class of, say, 1, one must understand sufficiently well the action of Möbius transformations on the solutions of the hypergeometric equation. This action

has also been analysed by Dwork in [Ku], a remarkable paper that adds also to the classical cohomological understanding of the Kummer transformations, and determines the effect of those transformations on the Frobenius matrix. We need to complement section 4 of [Ku] with a more flexible formula for the changes in the Frobenius matrix.

For simplicity, we will further assume in our calculations that the monodromy at 0 and 1 is semisimple as in [LDE, Chapter 25]: this assumption does not affect the final result. One can in fact similarly extend the results of [LDE, Chapter 26], to cover the case of logarithmic solutions. In any case, our method proves the Koblitz-Diamond formula under the slightly modified assumptions $\min(\mu_{a^{(i)}}, \mu_{b^{(i)}}) > 0$ and $\mu_{c^{(i)}} > \mu_{a^{(i)}} + \mu_{b^{(i)}}$ for any $i = 0, 1, \ldots$.

The main point of this article is computational. We realized that, while waiting for a more systematic exposition of the general ideas, recalled above, we were risking to forget about some difficult computations we had previously made to support our ideas. We therefore decided to store those computations in these proceedings, and at the same time to make them available to any willing colleague.

Contents.

1. Filtered logarithmic F-crystals.

1.1. Notation. Let K be a complete discrete valuation field of mixed characteristics $(0, p)$ with perfect residue field k, and let $\mathcal{V}$ denote its ring of integers. We denote by $|\ \ |$ the absolute value of K, normalized by $|p| = p^{-1}$. For simplicity we will assume here that $\mathcal{V} = W(k)$ is absolutely unramified, and we will denote by $\sigma : \mathcal{V} \to \mathcal{V}$ the Frobenius (and its extension to K).

1.2. Let X be a smooth p-adic formal scheme over of finite type over $\operatorname{Spf}\mathcal{V}$, and D be a divisor in X with strict normal crossings relative to $\mathcal{V}$. The pair (X, D) defines a fine log scheme (X, M_D) in the sense of Fontaine-Illusie and Kato [KK], over $\operatorname{Spf}\mathcal{V}$ endowed with the trivial log structure. It reduces modulo p to a similar pair (X_k, D_k). In local étale coordinates $(x_1, \ldots, x_n)$, we may assume that the ideal sheaf $\mathcal{I}_D$ of D is generated by $x_1 \cdots x_d$, and that the sheaf of log differentials $\Omega^1_X(\log D) = \Omega^1_{(X,D)}$ admits the $\mathcal{O}_X$-basis $\frac{dx_1}{x_1}, \ldots, \frac{dx_d}{x_d}, dx_{d+1}, \ldots, dx_n$. We denote by $\mathcal{S}_D$ the $\mathcal{O}_X$-algebra $\bigoplus_{j \geqslant 0} \mathcal{I}_D^{-j}$. For any formal scheme T over $\operatorname{Spf}\mathcal{V}$, T^σ will denote the formal scheme over $\operatorname{Spf}\mathcal{V}$ obtained by the base change σ. Similarly, for an $\mathcal{O}_T$-module $\mathcal{E}$ (with connection ∇), $\mathcal{E}^\sigma$ will denote the $\mathcal{O}_{T^\sigma}$-module (with connection ∇^σ) obtained by the base change σ.

1.3. Definitions. A *logarithmic crystal on* $(X, D)/\mathcal{V}$ consists of

(a) a finite projective $\mathcal{O}_X$-module $\mathcal{E}$;

(b) an integrable logarithmic connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes_A \Omega^1_X(\log D)$.

The logarithmic crystal $(\mathcal{E}, \nabla)$ on $(X, D)/\mathcal{V}$ is *convergent* if the following convergence condition holds:

(c) for any $\varepsilon > 0$, and any section e of $\mathcal{E}$ over a coordinate domain $(U, x_1, \ldots, x_n)$, the section of $\mathcal{E}$ defined by

$$p^{\varepsilon \Sigma_i \mu_i} \frac{1}{\Pi_i(\mu_i!)} \prod_{i=1}^{n} \prod_{j=0}^{\mu_i} (\nabla_i - j)e$$

converges to 0 when $\Sigma_i \mu_i$ goes to ∞. Here the operators ∇_i are defined by the formula $\nabla(e) = \sum_i \nabla_i(e) \frac{dx_i}{x_i}$.

A (resp. convergent) logarithmic crystal $(\mathcal{E}, \nabla)$ on $(X, D)/\mathcal{V}$ is *filtered* if it is equipped with

(d) a decreasing and finite filtration, exhaustive and separated, $\{Fil^i\mathcal{E}\}_i$ by local direct factors of $\mathcal{E}$ satisfying the Griffiths transversality $\nabla(Fil^i\mathcal{E}) \subseteq Fil^{i-1}\mathcal{E} \otimes \Omega^1_X(\log D)$.

For $f \in \mathbb{Z}_{\geqslant 1}$, a lifting of the f-th order relative Frobenius of X_k to the log scheme (X, D) is a lifting $\varphi_f : X \to X^{\sigma^f}$ of the f-th order relative Frobenius of X_k (coming from the f-th iterate of the standard Frobenius) *adapted* to the divisor D, *i.e.* such that $\varphi_f^* D^{\sigma^f} = p^f D$. So, in terms of local coordinates as above, for any i one has $\varphi_f^*(x_i) = x_i^{p^f}$ up to units in $\mathcal{O}_X$. When $f = 1$, we avoid mentioning f.

A *logarithmic F-crystal over* $(X, D)/\mathcal{V}$ for the f-th order Frobenius, is a convergent logarithmic crystal $(\mathcal{E}, \nabla)$ over $(X, D)/\mathcal{V}$, together with the assignment, for any local lifting of the f-th order Frobenius $\varphi_f : U \to U^{\sigma^f}$, adapted to $D \cap U$, to an open formal $\mathcal{V}$-subscheme U of X, of a horizontal $\mathcal{S}_{D\cap U}$-linear monomorphism

$$F(\varphi_f) : \varphi_f^*(\mathcal{S}_D \otimes \mathcal{E}, \nabla)^{\sigma^f}_{|U^{\sigma^f}} \to (\mathcal{S}_D \otimes \mathcal{E}, \nabla)_{|U}$$

which becomes an isomorphism when tensored with K. (Notice that ∇ induces a logarithmic connection on $\mathcal{S}_D \otimes \mathcal{E}$, and that φ_f extends to a morphism of ringed spaces $(U, \mathcal{S}_{D\cap U}) \to (U, \mathcal{S}_{D\cap U})^{\sigma^f}$.)

We say that $(\mathcal{E}, \{Fil^i\mathcal{E}\}_i, \nabla, F)$ is a *filtered logarithmic F-crystal over* $(X, D)/\mathcal{V}$ for the f-th order Frobenius, if $(\mathcal{E}, \nabla, F)$ is a logarithmic F-crystal over $(X, D)/\mathcal{V}$ for the f-th order Frobenius, and $(\mathcal{E}, \{Fil^i\mathcal{E}\}_i, \nabla)$ a filtered convergent logarithmic crystal over $(X, D)/\mathcal{V}$. Then, $(\mathcal{E}, \{Fil^i\mathcal{E}\}_i, \nabla, F)$ is *divisible* if, for any i,

$$F(\varphi_f)\left(\varphi_f^*(Fil^i\mathcal{E}^{\sigma^f})_{|U^{\sigma^f}}\right) \subset p^i(\mathcal{S}_D \otimes \mathcal{E})_{|U} \ .$$

(In our application, the stronger condition

$$F(\varphi_f)\left(\varphi_f^*(Fil^i\mathcal{E}^{\sigma^f})_{|U^{\sigma^f}}\right) \subset p^{fi}(\mathcal{S}_D \otimes \mathcal{E})_{|U} \ ,$$

will be considered, at the expense of the strength of our results.) We omit the natural extensions of the previous definitions to a relative situation $(X, D)/S$, where S is a formal p-adic scheme, smooth and of finite type over $\mathrm{Spf}\, \mathcal{V}$ and D is a divisor in X, with strict normal crossings relative to S. In the present discussion however a logarithmic crystal on $(X, D)/S$ will rarely be convergent in the natural relative generalization of this notion. This is because the points of S will play for us the role of variable "exponents of monodromy" rather than the one of rational parameters as in the theory of Picard-Fuchs equations.

1.4. Let $(\mathcal{E}, \nabla, F)$ be a logarithmic F-crystal over $(X, D)/\mathcal{V}$ for the f-th order Frobenius, and let us assume, for simplicity, that X is of relative dimension 1 over $\mathcal{V}$, with coordinate x, and that D consists of a finite set of $\mathcal{V}$-valued points $x = t_1, \dots, t_r$, with $t_i \in \mathcal{V}$. In particular, as a topological space, D consists of a finite set of k-valued points $\overline{x} = \overline{t}_1, \dots, \overline{t}_r$, for the reduced coordinate $\overline{x}$. We recall the *bounded Robba ring* $\mathcal{R}_{X,\overline{t}_i}$ of X at $\overline{t}_i$, which is the ring of Laurent series $\sum_{j\in\mathbb{Z}} a_j(x - t_i)^j$, with $a_j \in \mathcal{V}$, converging in some annulus $\varepsilon < |x - t_i| < 1$. Then, according to [Ke, §4], one can develop Dieudonné theory over $\mathcal{R}_{X,\overline{t}_i}$, and consequently define the notion of *special Newton polygon* of $(\mathcal{E}, \nabla, F)$ at $\overline{t}_i$, that is over $\mathcal{R}_{X,\overline{t}_i}$. So, combining this result with the classical theory of Newton polygons of F-crystals [Ka], we associate to $(\mathcal{E}, \nabla, F)$ a Newton polygon at each $\overline{k}$-valued point of the k-scheme X_k, where $\overline{k}$ denotes the algebraic closure of k. We will say that $(\mathcal{E}, \nabla, F)$ is a *unit root* logarithmic F-crystal over $(X, D)/\mathcal{V}$, if its Newton polygon is a horizontal segment at any $\overline{k}$-valued point of the k-scheme X_k.

1.5. We will give an example of the following reasonable generalization of [Ka, 4.1], whose proof does not seem to appear in the literature. We plan to give full details elsewhere.

Theorem. *Let $(\mathcal{E}, \{Fil^2\mathcal{E} = 0 \subset Fil^1\mathcal{E} \subset Fil^0\mathcal{E} = \mathcal{E}\}, \nabla, F)$ be a divisible F-crystal for the f-th order Frobenius on $(X,D)/\mathcal{V}$, with a two-step filtration. Assume X is of relative dimension 1 over $\mathcal{V}$, and that at every $\overline{k}$-valued point of X_k, its Newton polygon begins with a side of slope zero, always of the same length $\nu \geqslant 1$ (i.e., point by point, the unit root part has rank ν) and that $\mathcal{E}/Fil^1\mathcal{E}$ is of constant rank ν. Then there exists a logarithmic unit root F-sub-crystal $(\mathcal{U}, \nabla, F) \subset (\mathcal{E}, \nabla, F)$, whose underlying module $\mathcal{U}$ is transversal to $Fil^1\mathcal{E}$ (i.e. $\mathcal{E} = \mathcal{U} \oplus Fil^1\mathcal{E}$).*

2. Dwork families of logarithmic F-crystals.

This section is an abstract formulation of the theory of generalized hypergeometric functions of [GHF] and [Ad].

2.1. Let (X,D) be as in the previous section, and let $H = \mathrm{Spf}\,\mathbb{Z}_p\{a_1,\dots,a_r\}$, $(X_H, D_H) = (X,D) \times H$, with projections $p_X : X_H \longrightarrow X$ and $p_H : X_H \to H$. The coordinates $a_1,\dots,a_r$ will play a special role in what follows, together with a finite set of linear forms $\mathcal{L} = \{\ell_1(\vec{a}),\dots,\ell_N(\vec{a})\} \subset \mathbb{Z}[a_1,\dots,a_r]$.

We will assume that the system of inequalities $\ell_i(\vec{a}) \geqslant 0$, for $i = 1,\dots,N$, defines a rational polyhedral cone $C_{\mathcal{L}}$ of dimension r in $\mathbb{R}^r$, and that, for any i, $\ell_i(\vec{a}) = 0$ is a 1-codimensional face of $C_{\mathcal{L}}$, and $\ell_i(\mathbb{Z}^r) = \mathbb{Z}$. A *meromorphic function* on an open formal $\mathcal{V}$-subscheme of X_H will be assumed to have a finite set of polar hypersurfaces of the form $p_X^{-1}(D)$ and of the form $p_H^{-1}(\ell_i(\vec{a}) = j)$, for $i = 1,\dots,N$ and $j \in \mathbb{Z}$. So, in local coordinates $(x_1,\dots,x_n)$ over an open formal subscheme U of X, with $D \cap U = V(x_1 \cdots x_d)$, a meromorphic function g on $U \times H$ will be a quotient of a section $h \in \Gamma(U \times H, \mathcal{O}_{X_H})$, by an expression of the form $x_1^{u_1} \cdots x_d^{u_d} \prod_{i,j}(\ell_i(\vec{a}) - j)$. We will loosely talk about meromorphic structures on X_H in that sense.

2.2. Let $\mathcal{E}_0$ be a locally free $\mathcal{O}_X$-module of finite type, $\mathcal{E} = p_X^*\mathcal{E}_0$, $\{Fil^i\mathcal{E}\}_i$ be a filtration of $\mathcal{E}_0$ by local direct factors, and let ∇ be an integrable $(X_H, D_H)/H$-connection on $\mathcal{E}$, such that $(\mathcal{E}, \{Fil^i\mathcal{E}\}_i, \nabla)$ becomes a filtered logarithmic crystal over $(X_H, D_H)/H$. We will replace the relative convergence condition by the following weaker convergence condition

(c') For any $\vec{a} \in H(\mathbb{Q} \cap \mathbb{Z}_p)$ $(\mathcal{E}_{\vec{a}}, \{(Fil^i\mathcal{E})_{\vec{a}}\}_i, \nabla_{\vec{a}}) = \vec{a}^*(\mathcal{E}, \{Fil^i\mathcal{E}\}_i, \nabla)$ is a filtered convergent logarithmic crystal.

2.3. For $\vec{u} \in \mathbb{G}_a^r(\mathbb{Z})$, we denote by

$$\sigma_{\vec{u}} : H \longrightarrow H$$

the translation mapping $\vec{a} \longmapsto \vec{a} + \vec{u}$, inducing

$$\sigma_{\vec{u}} = \mathrm{id}_X \times \sigma_{\vec{u}} : X_H \longrightarrow X_H \ .$$

We assume the existence of horizontal meromorphic isomorphisms

$$b_{\vec{u}} : (\mathcal{E}, \nabla) \longrightarrow \sigma_{\vec{u}}^*(\mathcal{E}, \nabla)$$

such that for $\vec{u}, \vec{v} \in \mathbb{G}_a^r(\mathbb{Z})$

$$b_{\vec{u}+\vec{v}} = \sigma_{\vec{u}}^*(b_{\vec{v}}) \circ b_{\vec{u}} \ .$$

We will assume that for $\vec{\mu} \in \mathbb{Z}^r \cap C_{\mathcal{L}}$, the only poles of the map $b_{\vec{u}}$ be along D. To be more precise [GHF, Conjecture 6.3.1], [Ad, Thm. 8.1], we may consider the natural map of ringed spaces $q_X : (X_H, p_X^*\mathcal{S}_D) \to X_H$. Then, we will assume that, for $\vec{u} \in \mathbb{Z}^r \cap C_{\mathcal{L}}$, $b_{\vec{u}}$ is a honest morphism

$$b_{\vec{u}} : q_X^*(\mathcal{E}, \nabla) \longrightarrow q_X^*\sigma_{\vec{u}}^*(\mathcal{E}, \nabla) \ ,$$

and that $\det b_{\vec{u}}$ vanishes if and only if

$$\prod_{i=1}^{N} (\ell_i(\vec{a}))_{\ell_i(\vec{u})} = 0 \ .$$

2.4. For $\vec{\mu} \in \mathbb{Z}^r$ and $\rho = p^{-s}$, $s \in \mathbb{Z}_{\geqslant 0}$, we define $\mathbb{D}(-\vec{\mu},\rho)$ to be the formal counterpart of the closed analytic disk $D(-\vec{\mu},\rho) = \{\vec{a} \in \mathbb{C}_p^r \mid |a_i + \mu_i| \leqslant \rho, \forall i = 1,\dots,r\}$, that is the formal $\mathcal{V}$-subscheme of H

$$\mathbb{D}(-\vec{\mu},\rho) = \operatorname{Spf} \mathcal{V}\{a_1,\dots,a_r,b_1,\dots,b_r\}/(p^s b_1 - a_1 - \mu_1,\dots,p^s b_r - a_r - \mu_r) \ .$$

For $\rho < 1$, there are natural morphisms

$$\begin{aligned} \tau_{\vec{\mu}} : \mathbb{D}(-\vec{\mu},\rho) &\longrightarrow \mathbb{D}(\vec{0},p\rho) \\ \vec{a} &\longmapsto \frac{\vec{a}+\vec{\mu}}{p} \ . \end{aligned}$$

Notice that for any $\vec{\mu},\vec{u},\vec{v} \in \mathbb{G}_a^r(\mathbb{Z})$

$$\sigma_{\vec{u}} \circ \tau_{\vec{\mu}} = \tau_{p\vec{u}-\vec{v}+\vec{\mu}} \circ \sigma_{\vec{v}} \ .$$

Let U be an open formal subscheme of X and $\varphi : U \longrightarrow U^\sigma$ be a σ-linear lifting of Frobenius adapted to $U \cap D$. We consider the map

$$\begin{aligned} \varphi \times \tau_{\vec{\mu}} : U \times \mathbb{D}(-\vec{\mu},\rho) &\longrightarrow U^\sigma \times \mathbb{D}(\vec{0},p\rho) \\ (x,\vec{a}) &\longmapsto \left(\varphi(x), \frac{\vec{a}+\vec{\mu}}{p}\right) \ . \end{aligned}$$

We assume that for each (U,φ), $\rho \in p^{\mathbb{Z}_{<0}}$, and $\vec{\mu} \in \mathbb{Z}^r$, as before, there exists a *meromorphic* morphism of logarithmic crystals over $(U \times \mathbb{D}(-\vec{\mu},\rho), D \times \mathbb{D}(-\vec{\mu},\rho))/\mathbb{D}(-\vec{\mu},\rho)$

$$F(\varphi,\vec{\mu}) : (\varphi \times \tau_{\vec{\mu}})^*(\mathcal{E}^\sigma)_{|U \times \mathbb{D}(-\vec{\mu},\rho)} \longrightarrow \mathcal{E}_{|U \times \mathbb{D}(-\vec{\mu},\rho)} \ .$$

The previous data should satisfy

$$\sigma_{\vec{v}}^*(F(\varphi, p\vec{u}-\vec{v}+\vec{\mu})) \circ (\varphi \times \tau_{\vec{\mu}})^*(b_{\vec{u}}^\sigma) = b_{\vec{v}} \circ F(\varphi,\vec{\mu}) \ .$$

What this simply means is that the map $F(\varphi,\vec{\mu})$ may be coherently regarded as a system of maps

$$F(\varphi,\vec{\mu})_{(x,\vec{a})} = F(\vec{a},\vec{b};x,\varphi(x)) : \mathcal{E}^\sigma_{(\vec{b},\varphi(x))} \to \mathcal{E}_{(\vec{a},x)} \ ,$$

for any, say, $\mathcal{V}$-valued points $(x,\vec{a})$ of $U \times \mathbb{D}(-\vec{\mu},\rho)$ and $(\varphi(x),\vec{b})$ of $U^\sigma \times \mathbb{D}(\vec{0},p\rho)$, with $p\vec{b} - \vec{a} = \vec{\mu} \in \mathbb{Z}^r$. It will also be sometimes convenient to use the notation $F(\varphi;\vec{a},\vec{b};x)$ or $F(\varphi;\vec{a},\vec{b})$ for that map. From the viewpoint of p-adic convergence, our condition means that the matrix $\gamma^{(\varphi)}(\vec{a},\vec{b};x)$ expressing $F(\varphi;\vec{a},\vec{b};x)$ in terms of a global basis of $\mathcal{E}_0$, is p-adically meromorphic in the variables $(\vec{a},\vec{b};x)$, for *fixed* $\vec{\mu} = p\vec{b} - \vec{a}$ in $\mathbb{Z}^r$ and $\vec{b} \in D(\vec{0},1)$.

A more precise meromorphy condition, satisfied for generalized hypergeometric functions [GHF, 4.7.1], [Ad, 9.12], is that the polar locus of $F(\varphi,\vec{\mu})$ should be contained in the union of $p_X^{-1}(D)$ and of the zero locus of the determinant of the map $(\varphi \times \tau_{\vec{\mu}})^*(b_{\vec{v}})$, where $\vec{v} \in \mathbb{Z}^r \cap C_{\mathcal{L}}$ is such that

$$\mathbb{Z}^r \cap (\vec{v} - \frac{\vec{\mu}}{p} + C_{\mathcal{L}}) \subset C_{\mathcal{L}} \ .$$

In particular, [GHF, 6.13.2], [Ad, 9.14], if $\vec{\mu} \in \mathbb{Z}^r \cap C_{\mathcal{L}}$, and $\ell_i(\vec{\mu}) \leqslant p-1$, for any $i = 1,\dots,N$, then $F(\varphi,\vec{\mu})$ is a honest morphism

$$F(\varphi,\vec{\mu}) : ((\varphi \times \tau_{\vec{\mu}})^*(q_X^*\mathcal{E})^\sigma)_{|U \times \mathbb{D}(-\vec{\mu},\rho)} \longrightarrow (q_X^*\mathcal{E})_{|U \times \mathbb{D}(-\vec{\mu},\rho)} \ .$$

2.5. Definition. A set of data $(\mathcal{E}, \{Fil^i\mathcal{E}\}_i, \nabla, \{b_{\vec{u}}\}_{\vec{u}}, F)$ as before, will be called a *Dwork family of filtered convergent logarithmic F-crystals on X, parametrized by H, with set of singular forms $\mathcal{L}$, on $(X,D)/\mathcal{V}$.*

2.6. The notion of divisibility for Dwork family of filtered F-crystals $(\mathcal{E}, \{Fil^i\mathcal{E}\}_i, \nabla, \{b_{\vec{u}}\}_{\vec{u}}, F)$, is perhaps a little unexpected. It first requires that the filtration $\{Fil^i\mathcal{E}\}_i$ is constructed in the following way. Let $\mathcal{A} = \{\vec{\mu}\}$ be a set of representatives of $H(\mathbb{Z}_p)$ modulo p. Therefore $H(\mathbb{Z}_p)$ is the disjoint union of $\mathbb{D}(-\vec{\mu}, p^{-1})$, for $\vec{\mu} \in \mathcal{A}$. We will assume to be given a family, indexed by $\vec{\mu} \in \mathcal{A}$, of filtrations $\{Fil^i_{\vec{\mu}}\mathcal{E}_0\}_i$ of $\mathcal{E}_0$ by local direct factors. We assume that, for all i and $\vec{\mu}$, on $X \times \mathbb{D}(-\vec{\mu}, p^{-1})$, $Fil^i\mathcal{E}$ coincides with the inverse image via the first projection of $Fil^i_{\vec{\mu}}\mathcal{E}_0$.

The Dwork family will then said to be *divisible* if, for any i, any $\vec{\mu} \in \mathbb{Z}^r$, and any $\rho = p^{-s} < 1$,

$$F(\varphi, \vec{\mu}) \left(((\varphi \times \tau_{\vec{\mu}})^*(Fil^i\mathcal{E})^{\sigma})_{|U \times \mathbb{D}(-\vec{\mu},\rho)} \right) \subset p^i (q_X^*\mathcal{E})_{|U \times \mathbb{D}(-\vec{\mu},\rho)} \ .$$

2.7. Let $(\mathcal{E}, \{Fil^i\mathcal{E}\}_i, \nabla, \{b_{\vec{u}}\}_{\vec{u}}, F)$ be a Dwork family of logarithmic convergent filtered F-crystals on $(X,D)/\mathcal{V}$, parametrized by $H = \mathrm{Spf}\, \mathbb{Z}\{a_1, \dots, a_r\}$, with set of singular forms $\mathcal{L} = \{\ell_1, \dots, \ell_N\}$.

Now, for $\vec{a} \in \mathbb{Z}_p^r$, let us choose two sequences $\vec{a}^{[i]} \in \mathbb{Z}_p^r$ and $\vec{\mu}^{[i]} \in \mathbb{Z}^r$, $i = 0, 1, \dots$, so that $\vec{a}^{[0]} = \vec{a}$, and $p\vec{a}^{[i+1]} - \vec{a}^{[i]} = \vec{\mu}^{[i]}$, for any i. One possible choice is $\vec{a}^{[i]} = \vec{a}^{(i)} = (a_1^{(i)}, \dots, a_r^{(i)})$ and $\vec{\mu}^{[i]} = \vec{\mu}_{\vec{a}^{[i]}} = (\mu_{a_1^{(i)}}, \dots, \mu_{a_r^{(i)}})$, as defined in the introduction. If $\vec{a} \in (\mathbb{Q} \cap \mathbb{Z}_p)^r$, then there exists $f \in \mathbb{Z}_{\geqslant 1}$ such that $(p^f - 1)\vec{a} \in \mathbb{Z}^r$: the minimal such f will be called the *period* of $\vec{a}$, and we will say that $\vec{a}$ is *of finite period* f. So, if $\vec{a}$ is of finite period f, we may arrange the previous choices so that $\vec{a}^{[f]} = \vec{a}$.

If U is an open formal subscheme of X and φ is a lifting of Frobenius on U, adapted to $D \cap U$, the map

$$F(\varphi, \vec{\mu}^{[i]}) : (\varphi \times \tau_{\vec{\mu}^{[i]}})^*(\mathcal{E}^{\sigma})_{|U \times \mathbb{D}(-\vec{\mu}^{[i]},\rho)} \longrightarrow \mathcal{E}_{|U \times \mathbb{D}(-\vec{\mu}^{[i]},\rho)}$$

is represented, in terms of a basis of global sections $\vec{e}$ of $\mathcal{E}_0$ over U, by a matrix of functions meromorphic in $U \times \mathbb{D}(-\vec{\mu}^{[i]}, \rho)$. So, for $\vec{a} \in H(\mathcal{V})$, outside of a well-understood polar locus (*e.g.* if $\ell_i(\vec{a}) \notin \mathbb{Z}$, for $i = 1, \dots, N$), the previous map can be specialized to induce a meromorphic map, necessarily horizontal,

$$F(\varphi; \vec{a}^{[i]}, \vec{a}^{[i+1]}) : (\varphi^*(\mathcal{E}_{\vec{a}^{[i+1]}}, \nabla_{\vec{a}^{[i+1]}})^{\sigma})_{|U} \longrightarrow (\mathcal{E}_{\vec{a}^{[i]}}, \nabla_{\vec{a}^{[i]}})_{|U} \ .$$

If moreover $\vec{a}$ is in $H(\mathbb{Q} \cap \mathbb{Z}_p)$, say $\vec{a}^{[f]} = \vec{a}$, our assumptions imply that

$$(\mathcal{E}_{\vec{a}^{[f]}}, \nabla_{\vec{a}^{[f]}}) = (\mathcal{E}_{\vec{a}}, \nabla_{\vec{a}}) \ ,$$

and, outside of some polar locus (*e.g.* if $\ell_i(\vec{a}^{[j]}) \notin \mathbb{Z}$, for $i = 1, \dots, N$ and $j = 0, \dots, f-1$), that

$$F(\varphi; \vec{a}^{[f-1]}, \vec{a}^{[f]}) \circ F(\varphi; \vec{a}^{[f-2]}, \vec{a}^{[f-1]}) \circ \cdots\cdots \circ F(\varphi; \vec{a}^{[0]}, \vec{a}^{[1]})$$

is a meromorphic horizontal map

$$F(\varphi^f; \vec{a}) : (\varphi^f)^*(\mathcal{E}_{\vec{a}}, \nabla_{\vec{a}})^{\sigma^f}_{|U} \longrightarrow (\mathcal{E}_{\vec{a}}, \nabla_{\vec{a}})_{|U}.$$

The structure $(\mathcal{E}_{\vec{a}}, \nabla_{\vec{a}}, F(\,-\,; \vec{a}))$ is then a logarithmic F-crystal on $(X,D)/\mathcal{V}$ for the f-th iterate of the Frobenius, in the usual sense.

The convergence assumption now implies, by [Ka, 3.1.2], that the solutions of $(\mathcal{E}_{\vec{a}}, \nabla_{\vec{a}})$ at any rigid point x of the Raynaud generic fiber X_K of X, not in the open tube of D_k, converge in the open tube of radius 1 around x. Therefore, by [BC1], the holomorphic part of the solution matrix at a point of D_K, in the sense of the classical theory of regular singularities, converges in the open tube of D_k.

2.8. We now propose an extension of **1.5** for divisible Dwork families of logarithmic F-crystals. This paper gives an example of this situation in relative dimension 1.

Question. *Let (X,D) be as in* **1.2**. *Let $(\mathcal{E},\{Fil^2\mathcal{E}=0\subset Fil^1\mathcal{E}\subset Fil^0\mathcal{E}=\mathcal{E}\},\nabla,\{b_{\vec{u}}\}_{\vec{u}\in\mathbb{Z}^r},F)$ be a divisible Dwork family of logarithmic convergent filtered F-crystals on $(X,D)/\mathcal{V}$, parametrized by $H=\operatorname{Spf}\mathbb{Z}\{a_1,\dots,a_r\}$, with set of singular forms $\mathcal{L}=\{\ell_1,\dots,\ell_N\}$. Let us assume that, for any $\vec{a}\in H(\mathbb{Z}_p)$ of finite period f, such that $\ell_i(\vec{a})\notin\mathbb{Z}$, for $i=1,\dots,N$, the divisible logarithmic F-crystal $(\mathcal{E}_{\vec{a}},\{Fil^2\mathcal{E}_{\vec{a}}=0\subset Fil^1\mathcal{E}_{\vec{a}}\subset Fil^0\mathcal{E}_{\vec{a}}=\mathcal{E}_{\vec{a}}\},\nabla_{\vec{a}},F(\ -\ ,\vec{a}))$ over $(X,D)/\mathcal{V}$ for the f-th order Frobenius, admits a logarithmic unit root F-sub-crystal $(\mathcal{U}_{\vec{a}},\nabla_{\vec{a}},F(\ -\ ,\vec{a}))$, whose underlying module $\mathcal{U}_{\vec{a}}$ is transversal to $Fil^1\mathcal{E}_{\vec{a}}$ (i.e. $\mathcal{E}_{\vec{a}}=\mathcal{U}_{\vec{a}}\oplus Fil^1\mathcal{E}_{\vec{a}}$). Then does there exist a logarithmic sub-crystal $(\mathcal{U},\nabla,F)$ of $(\mathcal{E},\nabla)$ on $(X_H,D_H)/H$, stable under the map F, in the sense that for any (U,φ) as above, for any $\vec{\mu}\in\mathbb{Z}^r$, and any $\rho=p^{-s}<1$, F induces a meromorphic morphism of logarithmic crystals over $(U\times\mathbb{D}(-\vec{\mu},\rho),D\times\mathbb{D}(-\vec{\mu},\rho))/\mathbb{D}(-\vec{\mu},\rho)$*

$$F(\varphi,\vec{\mu}):(\varphi\times\tau_{\vec{\mu}})^*(\mathcal{U}^\sigma)_{|U\times\mathbb{D}(-\vec{\mu},\rho)}\longrightarrow\mathcal{U}_{|U\times\mathbb{D}(-\vec{\mu},\rho)}\ ,$$

whose underlying module $\mathcal{U}$ is transversal to $Fil^1\mathcal{E}$ (i.e. $\mathcal{E}=\mathcal{U}\oplus Fil^1\mathcal{E}$) and whose specialization at any $\vec{a}\in H(\mathbb{Q}\cap\mathbb{Z}_p)$ coincides with $(\mathcal{U}_{\vec{a}},\nabla_{\vec{a}},F(\ -\ ,\vec{a}))$?

We remark that the rank of the underlying modules $\mathcal{U}_{\vec{a}}$ may vary with the class of $\vec{a}$ mod p. We treat a simplified version of the previous question, where we restrict our parameters $\vec{a}$ to a bizarre subset $\mathcal{T}_2$ of $H(\mathbb{Z}_p)$, stable under the map $\vec{a}\longmapsto\vec{a}^{(1)}$, and such that the filtration $Fil^2\mathcal{E}_{\vec{a}}=0\subset Fil^1\mathcal{E}_{\vec{a}}\subset Fil^0\mathcal{E}_{\vec{a}}=\mathcal{E}_{\vec{a}}$ of $\mathcal{E}_{\vec{a}}$ is independent of $\vec{a}\in\mathcal{T}_2$.

3. The hypergeometric family.

A very interesting example of the preceding situation is connected with the hypergeometric system:

$$\frac{dY}{d\lambda}=YG_{\vec{a}}(\lambda),\qquad\qquad \vec{a}=(a_1,a_2,a_3)\in\mathbb{Z}_p^3$$

where

$$G_{\vec{a}}(\lambda)=\begin{pmatrix}-\dfrac{a_3}{\lambda} & \dfrac{a_3-a_1}{1-\lambda}\\ \dfrac{a_3-a_2}{\lambda} & \dfrac{a_1+a_2-a_3}{1-\lambda}\end{pmatrix}\ .$$

In this case, the relevant linear forms are

$$\mathcal{L}=\{\ell_1(\vec{a})=a_3-a_1\ ,\ \ \ell_2(\vec{a})=a_3-a_2\ ,\ \ \ell_3(\vec{a})=a_2\ ,\ \ \ell_4(\vec{a})=a_1\}\ .$$

We denote by $C_{\vec{a}}(z,\lambda)$ the matrix solution at $z\neq 0,1,\infty$ such that $C_{\vec{a}}(z,z)=\mathbb{I}_2$, the 2×2 identity matrix. When the entries of $\vec{a}$ are rational, that matrix converges if $|\lambda-z|<|z|\min(1,|1-z|)$ [LDE].

3.0.1. The F-crystal structure of the hypergeometric system is expressed by the following data. We take $\vec{a},\vec{b}\in\mathbb{Z}_p^3$, $p\vec{b}-\vec{a}=\vec{\mu}\in\mathbb{Z}^3$, φ any lifting of Frobenius on an open formal subscheme U of $\hat{\mathbb{P}}$, adapted to $D=\{0,1,\infty\}$. We obtain a matrix $\gamma^{(\varphi)}(\vec{a},\vec{b};\lambda)$ meromorphic in the variables $(\vec{a},\vec{b},\lambda)$, *fpr fixed* $\vec{\mu}$, such that:

$$C^\sigma_{\vec{b}}\,(\varphi(z),\varphi(\lambda))\,\gamma^{(\varphi)}(\vec{a},\vec{b};\lambda)^t=\gamma^{(\varphi)}(\vec{a},\vec{b};z)^tC_{\vec{a}}(z,\lambda)\ ,$$

whenever $C_{\vec{a}}(z,\lambda)$ converges; this holds, in particular, if $\vec{a}\in(\mathbb{Q}\cap\mathbb{Z}_p)^3$, $z\neq0,1,\infty$ and $|\lambda-z|<|z|\min(1,|1-z|)$. Notice that we transpose the Frobenius matrix $\gamma^{(\varphi)}(\vec{a},\vec{b};\lambda)$ in this formula in view of compatibility with the notation of Dwork in [Ku] and [GHF].

3.0.2. We assume in our calculations that the local monodromy semisimple.

In the singular classes we formally write the solutions in the form:

$$\begin{array}{ll}\begin{pmatrix}1&0\\0&\lambda^{-a_3}\end{pmatrix}U^{(0)}_{\vec{a}}(\lambda) & U^{(0)}_{\vec{a}}(0)=\begin{pmatrix}a_3-a_2&a_3\\1-a_3&0\end{pmatrix}\\ \begin{pmatrix}1&0\\0&(1-\lambda)^{a_3-a_1-a_2}\end{pmatrix}U^{(1)}_{\vec{a}}(\lambda) & U^{(1)}_{\vec{a}}(1)=\begin{pmatrix}a_1+a_2-a_3&a_1-a_3\\0&a_1+a_2-a_3-1\end{pmatrix}\\ \begin{pmatrix}\lambda^{-a_1}&0\\0&\lambda^{-a_2}\end{pmatrix}U^{(\infty)}_{\vec{a}}(\lambda) & U^{(\infty)}_{\vec{a}}(\infty)=\begin{pmatrix}a_3-a_2&a_3-a_1\\a_2-a_1+1&a_2-a_1+1\end{pmatrix}\ .\end{array}$$

Under suitable assumptions on $\vec{a}$, $U_{\vec{a}}^{(i)}(\lambda)$ is a holomorphic matrix on the residue class of $i \in \{0,1,\infty\}$. Let $U_{\vec{a}}^{(i)}(\lambda) = \begin{pmatrix} u_1^{(i)} & u_2^{(i)} \\ u_3^{(i)} & u_4^{(i)} \end{pmatrix}$ for $i \in \{0,1,\infty\}$. We know that

$$\begin{cases} u_1^{(0)} = (a_3 - a_2) F(a_1, a_2, a_3 + 1; \lambda) \\ u_2^{(0)} = a_3 F(a_1, a_2, a_3; \lambda) \\ u_3^{(0)} = (1 - a_3) F(a_2 - a_3, a_1 - a_3, 1 - a_3; \lambda) \\ u_4^{(0)} = (a_3 - a_1) \lambda F(1 + a_2 - a_3, 1 + a_1 - a_3, 2 - a_3; \lambda) \end{cases}$$

$$\begin{cases} u_1^{(1)} = (a_1 + a_2 - a_3) F(a_1, a_2, a_1 + a_2 - a_3; 1 - \lambda) \\ u_2^{(1)} = (a_1 - a_3) F(a_1, a_2, a_1 + a_2 - a_3 + 1; 1 - \lambda) \\ u_3^{(1)} = (a_3 - a_2)(1 - \lambda) F(a_3 - a_1 + 1, a_3 - a_2 + 1, a_3 - a_1 - a_2 + 2; 1 - \lambda) \\ u_4^{(1)} = (a_1 + a_2 - a_3 - 1) F(a_3 - a_1, a_3 - a_2, a_3 - a_1 - a_2 + 1; 1 - \lambda) \end{cases}$$

$$\begin{cases} u_1^{(\infty)} = (a_3 - a_2) F(a_1, a_1 - a_3, a_1 - a_2 + 1; \lambda^{-1}) \\ u_2^{(\infty)} = (a_3 - a_1) F(a_1, a_1 - a_3 + 1, a_1 - a_2 + 1; \lambda^{-1}) \\ u_3^{(\infty)} = (a_2 - a_1 + 1) F(a_2 - a_3, a_2, a_2 - a_1 + 1; \lambda^{-1}) \\ u_4^{(\infty)} = (a_2 - a_1 + 1) F(a_2 - a_3 + 1, a_2, a_2 - a_1 + 1; \lambda^{-1}) \end{cases}$$

(*cf.* [Po, §22], or, more precisely, [LDE, Lemma 24.1] for the solution at 0).

3.0.3. We denote by φ_i a lifting of Frobenius to a formal neighborhood U_i of $i \in \{0,1,\infty\}$, adapted to i. Since $\varphi_0(\lambda) = \lambda^p\,(1 + \varepsilon\lambda u(\lambda))$ with $|\varepsilon| < 1$ and $u(\lambda) \in \mathcal{V}[\![\lambda]\!]$, we have that

$$\frac{\varphi_0(\lambda)^{b_3}}{\lambda^{a_3}} = \lambda^{\mu_3}\,(1 + \varepsilon\lambda v(\lambda))$$

where $v(\lambda) \in \mathcal{V}[\![\lambda]\!]$ converges inside a disk of radius > 1, and

$$U_{\vec{b}}^{(0)\,\sigma}\,(\varphi_0(\lambda))\,\gamma^{(\varphi_0)}(\vec{a},\vec{b};\lambda)^t = \begin{pmatrix} \xi_1^{(0)}(\vec{a},\vec{b}) & 0 \\ 0 & \xi_2^{(0)}(\vec{a},\vec{b})\dfrac{\varphi_0(\lambda)^{b_3}}{\lambda^{a_3}} \end{pmatrix} U_{\vec{a}}^{(0)}(\lambda)\ .$$

Similarly,

$$U_{\vec{b}}^{(1)\,\sigma}\,(\varphi_1(\lambda))\,\gamma^{(\varphi_1)}(\vec{a},\vec{b};\lambda)^t = \begin{pmatrix} \xi_1^{(1)}(\vec{a},\vec{b}) & 0 \\ 0 & \xi_2^{(1)}(\vec{a},\vec{b})\dfrac{(1-\varphi_1(\lambda))^{b_1+b_2-b_3}}{(1-\lambda)^{a_1+a_2-a_3}} \end{pmatrix} U_{\vec{a}}^{(1)}(\lambda)$$

and

$$U_{\vec{b}}^{(\infty)\,\sigma}\,(\varphi_\infty(\lambda))\,\gamma^{(\varphi_\infty)}(\vec{a},\vec{b};\lambda)^t = \begin{pmatrix} \xi_1^{(\infty)}(\vec{a},\vec{b})\dfrac{\lambda^{a_1}}{\varphi_\infty(\lambda)^{b_1}} & 0 \\ 0 & \xi_2^{(\infty)}(\vec{a},\vec{b})\dfrac{\lambda^{a_2}}{\varphi_\infty(\lambda)^{b_2}} \end{pmatrix} U_{\vec{a}}^{(\infty)}(\lambda)\ .$$

Inspection of the dominant terms at $\lambda = 0, 1, \infty$, makes it clear that, for any $i = 0, 1, \infty$ and $j = 1, 2$, the functions $\xi_j^{(i)}(\vec{a},\vec{b})$ have the same p-adic meromorphic behavior as the entries of the function $\gamma^{(\varphi_i)}(\vec{a},\vec{b};\lambda)$, with possibly some extra poles. This might be surprising, since the matrix $U_{\vec{a}}^{(i)}(\lambda)$ itself is *not* a p-adic meromorphic function of $(\vec{a},\lambda)$.

3.1. Determination of Frobenius matrix.

3.1.1. For the calculations we use notation and results of [Ku]. In that paper, Dwork investigates the effect of Kummer transformations on the solutions of the hypergeometric equation.

We need a generalized form of the theorem in [Ku, §4], where only the standard lifting of Frobenius $\lambda \mapsto \lambda^p$ is considered. We rewrite the original statement as

$$H_m(\vec{a},\lambda)\gamma\left(M_m(\vec{a}), M_m(\vec{b}); \vartheta_m(\lambda), \vartheta_m(\lambda^p)\right) = \gamma(\vec{a},\vec{b},\lambda)H^\sigma_m(\vec{b},\lambda^p)$$

(this is possible once we make explicit $h_m(\vec{a},\vec{b},\lambda) = h_m(\vec{a},\lambda)/h^\sigma_m(\vec{b},\lambda^p)$ and $H_m(\vec{a},\lambda) = h_m(\vec{a},\lambda)N_m(\lambda)$ in the original formula

$$h_m(\vec{a},\vec{b},\lambda)\gamma\left(M_m(\vec{a}), M_m(\vec{b}); \vartheta_m(\lambda), \vartheta_m(\lambda^p)\right) = N_m(\lambda)^{-1}\gamma(\vec{a},\vec{b},\lambda)N_m(\lambda^p)$$

of Dwork's article). Our generalized statement is then

Theorem. *Let $\lambda \mapsto \varphi(\lambda)$ be a function analytic in a region of the type*

$$\inf\left\{|\lambda|, |\lambda^{-1}|, |1-\lambda|\right\} > 1-\varepsilon$$

for some $\varepsilon > 0$ and "close to Frobenius", in the sense that

$$|\varphi(\lambda) - \lambda^p| < 1$$

for every λ for which $|\lambda| = 1 = |1-\lambda|$. Then

$$H_m(\vec{a},\lambda)\gamma\left(M_m(\vec{a}), M_m(\vec{b}); \vartheta_m(\lambda), \vartheta_m(\varphi(\lambda))\right) = \gamma(\vec{a},\vec{b};\lambda,\varphi(\lambda))H^\sigma_m(\vec{b},\varphi(\lambda))$$

We also write the previous formula in the form

$$H_m(\vec{a},\lambda)\gamma^{(\varphi)}\left(M_m(\vec{a}), M_m(\vec{b}); \vartheta_m(\lambda)\right) = \gamma^{(\varphi)}(\vec{a},\vec{b};\lambda)H^\sigma_m(\vec{b},\varphi(\lambda)) \ .$$

Proof. We first use formula (3.1.7) of [Ku] for the variation of the lifting of Frobenius, we then apply the original statement, then formula (2.9) of [Ku] (behaviour of solutions under Kummer transformations) and finally again (3.1.7) of [Ku]:

$$\begin{aligned}
&H_m(\vec{a},\lambda)\gamma^{(\varphi)}\left(M_m(\vec{a}), M_m(\vec{b}); \vartheta_m(\lambda)\right) = \\
&= H_m(\vec{a},\lambda)\gamma^{(\lambda\mapsto\lambda^p)}\left(M_m(\vec{a}), M_m(\vec{b}); \vartheta_m(\lambda)\right) C_{M_m(\vec{b})}\left(\vartheta_m(\varphi(\lambda)), \vartheta_m(\lambda^p)\right)^t = \\
&= \gamma^{(\lambda\mapsto\lambda^p)}(\vec{a},\vec{b};\lambda)H^\sigma_m(\vec{b},\lambda^p)C_{M_m(\vec{b})}\left(\vartheta_m(\varphi(\lambda)), \vartheta_m(\lambda^p)\right)^t = \\
&= \gamma^{(\lambda\mapsto\lambda^p)}(\vec{a},\vec{b};\lambda)C_{\vec{b}}(\varphi(\lambda),\lambda^p)^t H^\sigma_m(\vec{b},\varphi(\lambda)) = \\
&= \gamma^{(\varphi)}(\vec{a},\vec{b};\lambda)H^\sigma_m(\vec{b},\varphi(\lambda)) \ .
\end{aligned}$$

□

3.1.2. Notation. In the following formulas we use Dwork's symbol $\gamma_p(x,y)$ defined in [LDE, ch. 21], for $py - x = \mu \in \mathbb{Z}$ and $d(x,\mathbb{Z}) \leqslant p^{-1}$, by

$$\gamma_p(x+m, y+n) = (-\pi)^{n-m}\frac{\Gamma(x+m)\Gamma(y)}{\Gamma(x)\Gamma(y+n)}\gamma_p(x,y)$$

for any $m, n \in \mathbb{Z}$ and

$$\gamma_p(x,y) = \pi^\mu \Gamma_p(x)$$

if $\mu \in \{0, 1, \dots, p-1\}$. Here π is a fixed element in $\overline{\mathbb{Q}}_p$ such that $\pi^{p-1} = -p$. We recall also the symplectic relation

$$\gamma_p(x,y)\gamma_p(1-x, 1-y) = (-)^{py-x}p$$

(equivalent to $\Gamma_p(x)\Gamma_p(1-x) = -(-)^t$ with $t \equiv -x \bmod p$, $t \in \{0, \dots, p-1\}$).

3.1.3. For the Frobenius matrix at the origin, Dwork [LDE, ch. 25] obtains the values:

$$\begin{aligned}
\xi^{(0)}_1(\vec{a},\vec{b}) &= \frac{\gamma_p(a_2,b_2)\gamma_p(a_3-a_2, b_3-b_2)}{\gamma_p(1+a_3, 1+b_3)} \\
\xi^{(0)}_2(\vec{a},\vec{b}) &= (-)^{\mu_2-\mu_3}\frac{\gamma_p(a_3-1, b_3-1)\gamma_p(1-a_1, 1-b_1)}{\gamma_p(1+a_3-a_1, 1+b_3-b_1)} \ .
\end{aligned}$$

We complete his calculations:

3.1.4. Theorem.

$$\xi_j^{(1)}(\vec{a},\vec{b}) = (-)^{\mu_2}\xi_j^{(0)}\left(M^{(1)}(\vec{a}), M^{(1)}(\vec{b})\right)$$
$$\xi_j^{(\infty)}(\vec{a},\vec{b}) = (-)^{\mu_1+\mu_2-\mu_3}\xi_j^{(0)}\left(M^{(\infty)}(\vec{a}), M^{(\infty)}(\vec{b})\right)$$

where $j = 1, 2$ and

$$M^{(1)}(\vec{a}) = (a_1, a_2, a_1 + a_2 - a_3)$$
$$M^{(\infty)}(\vec{a}) = (a_1, a_1 - a_3, a_1 - a_2)\ .$$

Proof. The argument is to compare the Frobenius action at infinity with the Frobenius action at the origin by using the transformation ϑ_9 of [Ku]; then to compare the Frobenius at 1 with the one at infinity by using the transformation ϑ_{11} of [Ku]. We need for a lemma relating the analytic part of solutions at a point and its image:

3.1.5. Lemma.

$$U_{\vec{a}}^{(\infty)}(\lambda) = U_{M_9(\vec{a})}^{(0)}(\vartheta_9(\lambda))N_9^t$$
$$U_{\vec{a}}^{(1)}(\lambda) = U_{M_5(\vec{a})}^{(0)}(\vartheta_5(\lambda))N_5^t$$
$$U_{\vec{a}}^{(1)}(\lambda) = U_{M_{11}(\vec{a})}^{(\infty)}(\vartheta_{11}(\lambda))N_{11}^t$$

where the transformations are named after Dwork's tables in [Ku]. In particular $\vartheta_9(\lambda) = \lambda^{-1}$, $\vartheta_5(\lambda) = 1 - \lambda$, $\vartheta_{11}(\lambda) = (1 - \lambda)^{-1}$.

Proof (Lemma). For convenience we report the essential information about the transformations we need, from Dwork's tables in [Ku]:

$$\begin{cases} \vartheta_9(\lambda) = \lambda^{-1} & M_9(\vec{a}) = (a_1, a_1 - a_3, a_1 - a_2) \\ h_9(\vec{a},\lambda) = (-)^{a_3-a_1-a_2}\lambda^{-a_1} & N_9(\vec{a}) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \\ h_9(\vec{a},\vec{b},\lambda) = (-)^{\mu_1+\mu_2-\mu_3}\lambda^{-\mu_1} & \end{cases}$$
$$\begin{cases} \vartheta_5(\lambda) = 1 - \lambda & M_5(\vec{a}) = (a_1, a_2, a_1 + a_2 - a_3) \\ h_5(\vec{a},\lambda) = (-)^{a_2} & N_5(\vec{a}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ h_5(\vec{a},\vec{b},\lambda) = (-)^{-\mu_2} & \end{cases}$$
$$\begin{cases} \vartheta_{11}(\lambda) = (1 - \lambda)^{-1} & M_{11}(\vec{a}) = (a_1, a_3 - a_2, a_1 - a_2) \\ h_{11}(\vec{a},\lambda) = (-)^{a_3-a_2}(1-\lambda)^{-a_1} & N_{11}(\vec{a}) = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \\ h_{11}(\vec{a},\vec{b},\lambda) = (-)^{\mu_2-\mu_3}\frac{(1-\lambda^p)^{b_1}}{(1-\lambda)^{a_1}} & \end{cases}$$

Moreover $H_m(\vec{a},\lambda) = h_m(\vec{a},\lambda)N_m(\vec{a})$.

From [Ku, §2], for a solution matrix $C_{\vec{a}}(x,\lambda)$ of the hypergeometric system at a point x, then $C_{M_m(\vec{a})}(x,\vartheta_m(\lambda))H_m(\vec{a},\lambda)^t$ is a solution matrix of the hypergeometric equation at $\vartheta_m(x)$. So

$$\begin{pmatrix} 1 & 0 \\ 0 & \vartheta_9(\lambda)^{-M_9(\vec{a})_3} \end{pmatrix} U_{M_9(\vec{a})}^{(0)}(\vartheta_9(\lambda))H_9(\vec{a},\lambda)^t = \begin{pmatrix} \lambda^{-a_1} & 0 \\ 0 & \lambda^{-a_2} \end{pmatrix} U_{M_9(\vec{a})}^{(0)}(\vartheta_9(\lambda))N_9^t$$

is a solution at infinity; comparing with the solution in **3.0.2** we obtain the first formula.

For the second we start with

$$\begin{pmatrix} 1 & 0 \\ 0 & \vartheta_5(\lambda)^{-M_5(\vec{a})_3} \end{pmatrix} U_{M_5(\vec{a})}^{(0)}(\vartheta_5(\lambda))H_5(\vec{a},\lambda)^t = \ = \begin{pmatrix} 1 & 0 \\ 0 & (1-\lambda)^{a_3-a_1-a_2} \end{pmatrix} U_{M_5(\vec{a})}^{(0)}(\vartheta_9(\lambda))N_5^t\ ,$$

a solution matrix at 1, and compare it with the given solution at that point. For the third we begin with

$$\begin{pmatrix} \vartheta_{11}(\lambda)^{-M_{11}(\vec{a})_1} & 0 \\ 0 & \vartheta_{11}(\lambda)^{M_{11}(\vec{a})_2-M_{11}(\vec{a})_3} \end{pmatrix} U^{(\infty)}_{M_{11}(\vec{a})}(\vartheta_{11}(\lambda))H_{11}(\vec{a},\lambda)^t =$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & (1-\lambda)^{a_3-a_1-a_2} \end{pmatrix} U^{(\infty)}_{M_{11}(\vec{a})}(\vartheta_{11}(\lambda))N^t_{11}$$

and use the same argument. □

We now come to the proof of our theorem. For the second formula, we use the standard lifting of Frobenius, adapted to the origin and to infinity, $\lambda \mapsto \lambda^p$, to write

$$U^{(\infty)\,\sigma}_{\vec{b}}(\lambda^p)\,\gamma(\vec{a},\vec{b};\lambda)^t = \begin{pmatrix} \xi^{(\infty)}_1(\vec{a},\vec{b})\lambda^{\mu_1} & 0 \\ 0 & \xi^{(\infty)}_2(\vec{a},\vec{b})\lambda^{\mu_2} \end{pmatrix} U^{(\infty)}_{\vec{a}}(\lambda) \ .$$

Using the lemma and the following formula of [Ku §4]:

$$\gamma(\vec{a},\vec{b};\lambda)^t = h_m(\vec{a},\vec{b};\lambda)N^*_m\gamma\left(M_m(\vec{a}),M_m(\vec{b});\vartheta_m(\lambda)\right)^t N^t_m \ ,$$

for $m=9$ and $h_9(\vec{a},\vec{b};\lambda) = (-)^{\mu_1+\mu_2-\mu_3}\lambda^{\mu_1}$, we obtain

$$(\infty') \qquad \begin{aligned} &U^{(0)\,\sigma}_{M_9(\vec{b})}\left(\frac{1}{\lambda^p}\right)\gamma\left(M_9(\vec{a}),M_9(\vec{b});\frac{1}{\lambda}\right)^t = \\ &= (-)^{\mu_1+\mu_2-\mu_3}\begin{pmatrix} \xi^{(\infty)}_1(\vec{a},\vec{b}) & 0 \\ 0 & \xi^{(\infty)}_2(\vec{a},\vec{b})\lambda^{\mu_2-\mu_1} \end{pmatrix} U^{(0)}_{M_9(\vec{a})}\left(\frac{1}{\lambda}\right) \ . \end{aligned}$$

On the other side, from the solution at the origin, by using $M_9(\vec{a})$ instead of $\vec{a}$ and $1/\lambda$ instead of λ, we can write:

$$(\infty'') \qquad \begin{aligned} &U^{(0)\,\sigma}_{M_9(\vec{b})}\left(\frac{1}{\lambda^p}\right)\gamma\left(M_9(\vec{a}),M_9(\vec{b});\frac{1}{\lambda}\right)^t = \\ &= \begin{pmatrix} \xi^{(0)}_1(M_9(\vec{a}),M_9(\vec{b})) & 0 \\ 0 & \xi^{(0)}_2(M_9(\vec{a}),M_9(\vec{b}))\lambda^{-M_9(\vec{\mu})_3} \end{pmatrix} U^{(0)}_{M_9(\vec{a})}\left(\frac{1}{\lambda}\right) \ . \end{aligned}$$

Comparing (∞') with (∞''), and defining $M^{(\infty)} = M_9$, we conclude.

For the first formula of the theorem, we consider the lifting of Frobenius, adapted to one and infinity, $\lambda \mapsto \varphi(\lambda) = 1-(1-\lambda)^p$, and we write

$$U^{(1)\,\sigma}_{\vec{b}}(\varphi(\lambda))\,\gamma^{(\varphi)}(\vec{a},\vec{b};\lambda)^t = \begin{pmatrix} \xi^{(1)}_1(\vec{a},\vec{b}) & 0 \\ 0 & \xi^{(1)}_2(\vec{a},\vec{b})(1-\lambda)^{\mu_1+\mu_2-\mu_3} \end{pmatrix} U^{(1)}_{\vec{a}}(\lambda) \ .$$

By the lemma, and replacing $(1-\lambda)^{-1}$ by λ, we have

$$U^{(\infty)\,\sigma}_{M_{11}(\vec{b})}(\lambda^p)\,N^t_{11}\gamma^{(\varphi)}\left(\vec{a},\vec{b};1-\frac{1}{\lambda}\right)^t = \begin{pmatrix} \xi^{(1)}_1(\vec{a},\vec{b}) & 0 \\ 0 & \xi^{(1)}_2(\vec{a},\vec{b})\lambda^{\mu_3-\mu_1-\mu_2} \end{pmatrix} U^{(\infty)}_{M_{11}(\vec{a})}(\lambda)N^t_{11} \ .$$

Now, the transformation ϑ_7 is inverse to ϑ_{11}. We use again the formula of [Ku §4], specialized to

$$\gamma^{(\varphi)}(\vec{a},\vec{b};\vartheta_7(\lambda))^t = h_7(M_{11}(\vec{a}),M_{11}(\vec{b});\lambda)^{-1}N^t_7\gamma^{(\varphi)}(M_{11}(\vec{a}),M_{11}(\vec{b});\lambda)^t N^*_7$$

with $h_7(M_{11}(\vec{a}),M_{11}(\vec{b});\lambda) = (-)^{-M_{11}(\vec{\mu})_2}\lambda^{M_{11}(\vec{\mu})_1} = (-)^{\mu_3-\mu_2}\lambda^{\mu_1}$, to obtain

$$(1') \qquad \begin{aligned} &U^{(\infty)\,\sigma}_{M_{11}(\vec{b})}(\lambda^p)\,\gamma^{(\varphi)}(M_{11}(\vec{a}),M_{11}(\vec{b});\lambda)^t = \\ &= (-)^{\mu_2-\mu_3}\begin{pmatrix} \xi^{(1)}_1(\vec{a},\vec{b})\lambda^{\mu_1} & 0 \\ 0 & \xi^{(1)}_2(\vec{a},\vec{b})\lambda^{\mu_3-\mu_2} \end{pmatrix} U^{(\infty)}_{M_{11}(\vec{a})}(\lambda) \ . \end{aligned}$$

On the other side, using the solution at infinity, and $M_{11}(\vec{a})$ instead of $\vec{a}$, we can write:

$$
\begin{aligned}
&U^{(\infty)\,\sigma}_{M_{11}(\vec{b})}(\lambda^p)\,\gamma^{(\varphi)}(M_{11}(\vec{a}),M_{11}(\vec{b});\lambda)^t = \\
(1'') \qquad &= \begin{pmatrix} \xi_1^{(\infty)}(M_{11}(\vec{a}),M_{11}(\vec{b}))\lambda^{M_{11}(\vec{\mu})_1} & 0 \\ 0 & \xi_2^{(\infty)}(M_{11}(\vec{a}),M_{11}(\vec{b}))\lambda^{M_{11}(\vec{\mu})_2} \end{pmatrix} U^{(\infty)}_{M_{11}(\vec{a})}(\lambda)\ .
\end{aligned}
$$

Again, comparing $(1')$ and $(1'')$ gives the equalities, for $j = 1, 2$,

$$\xi_j^{(1)}(\vec{a},\vec{b}) = (-)^{\mu_2-\mu_3}\xi_j^{(\infty)}\left(M_{11}(\vec{a}),M_{11}(\vec{b})\right)\ .$$

Finally, using the formulae for $\xi_j^{(\infty)}$, we obtain the results, where we define $M^{(1)} := M_5 = M_9 \circ M_{11}$.
□

3.2. The unit root subcrystal.
We know that, under the conditions $\vec{\mu} \in \{0, 1, \dots, p-1\}^3$ and

$$(1) \qquad \mu_3 < \min(\mu_1, \mu_2)$$

or

$$(2) \qquad \mu_3 > \max(\mu_1, \mu_2)\,,$$

for any lifting Frobenius φ the matrix $\gamma^{(\varphi)}(\vec{a},\vec{b};\lambda)$ is of the form

$$(1) \qquad \begin{pmatrix} A & B \\ pC & pD \end{pmatrix}$$

or

$$(2) \qquad \begin{pmatrix} pA & pB \\ C & D \end{pmatrix},$$

respectively, where A, B, C, D are analytic functions, bounded by 1 on a domain of the form $D(-\vec{\mu}, p^{-1}) \times \mathcal{S}_\varepsilon$ where

$$\mathcal{S}_\varepsilon = D(0, \varepsilon^{-1}) \setminus (D(0,\varepsilon) \cup D(1,\varepsilon))$$

for $\varepsilon \in (0, 1)$.

Moreover, for each $\vec{\mu}$ as before, there is a polynomial $F_{\vec{\mu}}(\lambda) \in \mathbb{Z}[\lambda]$, of degree $p-1$, such that the region $|F_{\vec{\mu}}(\lambda)| < 1$, consists of $p-1$ residue classes, called *supersingular* with the property that, for any hypergeometric system with parameters $\vec{a} \in D(-\vec{\mu}, p^{-1})$, on all classes $D(z, 1^-)$ not singular nor supersingular, one has $|A(z)| = 1$ (resp. $|D(z)| = 1$) in case (1) (resp. (2)).

In case (2) we define

$$
\begin{aligned}
\mathcal{T}_2 &= \left\{\vec{a} \in \mathbb{Z}_p^3 \mid \mu_{a_3^{(i)}} > \max(\mu_{a_1^{(i)}}, \mu_{a_2^{(i)}}) \forall i\right\} \\
H_2 &= \left\{\lambda \in \mathbb{P}^{1\,rig}_{\mathbb{C}_p} \mid \exists \vec{a} \in \mathcal{T}_2 \ s.t.\ |F_{\vec{\mu}}(\lambda)| < 1\right\} \\
\mathcal{S}_2 &= \mathbb{P}^{1\,rig}_{\mathbb{C}_p} \setminus (\text{singular classes} \cup H_2)\ .
\end{aligned}
$$

So $\mathcal{S}_2$ is the complement of a finite number of residue classes, and, for any lifting of Frobenius $\varphi:\ \mathcal{S}_2 \to \mathcal{S}_2$ we consider the map

$$
\begin{array}{rccc}
\overline{\varphi}: & \mathcal{T}_2 \times \mathcal{S}_2 & \longrightarrow & \mathcal{T}_2 \times \mathcal{S}_2 \\
& (\vec{a}, \lambda) & \longmapsto & (\vec{a}', \varphi(\lambda))\ .
\end{array}
$$

The search for the unit root F-subcrystal entails to write the bounded solution inside a residue class $D(z,1^-)\subset\mathcal{S}_2$ as $(\eta u,u)$. Such a solution is an eigenvector of Frobenius with a unit eigenvalue:

$$\overline{\varphi}^*(\eta u,u)\gamma^{(\varphi)}=\xi(\eta u,u)$$

with $|\xi|=1$. So we have

$$\eta=\frac{pA\overline{\varphi}^*(\eta)+C}{pB\overline{\varphi}^*(\eta)+D}$$

and this proves the analyticity of η in $\mathcal{T}_2\times\mathcal{S}_2$, because the function

$$(\mathcal{M})\qquad\qquad \omega\longmapsto\frac{pA\overline{\varphi}^*(\omega)+C}{pB\overline{\varphi}^*(\omega)+D}$$

is a contraction of the Banach space of analytic functions bounded by 1 on $\mathcal{T}_2\times\mathcal{S}_2$.

The unit root F-subcrystal is then defined over $\mathcal{T}_2\times\mathcal{S}_2$ by

$$\frac{u'}{u}=\frac{a_3-a_1}{1-\lambda}\eta+\frac{a_1+a_2-a_3}{1-\lambda}\ .$$

Our main result is that, if φ is adapted at the singular point $z\in\{0,1,\infty\}$, then the map $\mathcal{M}$ is a contraction of the space of analytic functions bounded by 1 on $\mathcal{T}_2\times D(z,1^-)$. Therefore η admits an extension inside the three singular classes (unless they are at the same time supersingular!) and therefore the unit root F-subcrystal also extends as a logarithmic F-subcrystal of the hypergeometric system everywhere except on the supersingular locus. Moreover, this logarithmic F-subcrystal is not singular (*i.e.* it is an F-crystal in the usual sense) in the class $D(z,1^-)$ for $z\in\{0,1,\infty\}$ if and only if the bounded solution is holomorphic, that is, if and only if $\left|\xi_1^{(z)}\right|=1$.

3.3. The Koblitz-Diamond formula.

Under the hypothesis $\vec{a}\in\mathcal{T}_2$ we have $|\xi_1^{(0)}(\vec{a},\vec{a}')|=1$, so the first row of $U_{\vec{a}}^{(0)}$ is bounded by 1, *i.e.* the unit F-subcrystal is not singular in the class $D(0,1^-)$. The hypothesis of the Koblitz-Diamond theorem implies in fact that the same is true in the class $D(1,1^-)$. Then under these hypotheses the unit root F-subcrystal is a crystal in the usual sense in a region containing both classes $D(0,1^-)$ and $D(1,1^-)$. The bounded solution in these two classes is $\left(u_1^{(z)},u_2^{(z)}\right)$ and $\eta=\dfrac{u_1^{(z)}}{u_2^{(z)}}$.

Let φ be a lifting of Frobenius to the formal affine line $\hat{\mathbb{A}}$, adapted at 0 and 1: for example one could take $\varphi\left(\frac{\lambda}{1-\lambda}\right)=\left(\frac{\lambda}{1-\lambda}\right)^p$. We note that, if ϑ indicates the transformation $\lambda\mapsto\dfrac{\lambda}{1-\lambda}$, one obtains

$$\lambda=\vartheta^{-1}\left(\vartheta(\lambda)\right)=\frac{t}{1+t}\circ\frac{\lambda}{1-\lambda}$$

therefore

$$\varphi(\lambda)=\vartheta^{-1}\left(\vartheta(\lambda)^p\right)=\frac{\lambda^p/(1-\lambda)^p}{1+\lambda^p/(1-\lambda)^p}=\frac{\lambda^p}{(1-\lambda)^p+\lambda^p}=\lambda^p\big(1-p\lambda P(\lambda)\big)^{-1}$$

where $P(\lambda)=\sum\limits_{i=1}^{p-1}\dfrac{1}{p}\dbinom{p}{i}(-\lambda)^{i-1}$, hence $\varphi(\lambda)\in\mathcal{O}\left(\hat{\mathbb{A}}\right)$. In similar way we obtain

$$\varphi(1-\lambda)=(1-\lambda)^p\big(1-p(1-\lambda)P(1-\lambda)\big)^{-1}.$$

From the formulas

$$\overline{\varphi}^*\left(u_1^{(z)},u_2^{(z)}\right)\gamma^{(\varphi)}=\xi_1^{(z)}\left(u_1^{(z)},u_2^{(z)}\right)$$

for $z \in \{0,1\}$, we obtain

$$\overline{\varphi}^*\left(u_1^{(z)}\right)\gamma_{12}^{(\varphi)}+\overline{\varphi}^*\left(u_2^{(z)}\right)\gamma_{22}^{(\varphi)}=\xi_1^{(z)}u_2^{(z)}$$

and (dividing by $\overline{\varphi}^*\left(u_2^{(z)}\right)$)

$$\overline{\varphi}^*(\eta)\gamma_{12}^{(\varphi)}+\gamma_{22}^{(\varphi)}=\xi_1^{(z)}\frac{u_2^{(z)}}{\overline{\varphi}^*\left(u_2^{(z)}\right)}$$

for $z \in \{0,1\}$. Then, if φ is adapted to 0 and 1, we have the equality

$$\xi_1^{(0)}(\vec{a},\vec{a}')\frac{a_3}{a_3'}\mathcal{F}^{(\varphi)}(\vec{a};\lambda)=\xi_1^{(1)}(\vec{a},\vec{a}')\frac{a_1-a_3}{a_1'-a_3'}\mathcal{F}^{(\varphi)}(a_1,a_2,a_1+a_2-a_3+1;1-\lambda)$$

as analytic functions on $\mathcal{T}_2\times(\mathcal{S}_2\cup D(0,1^-)\cup D(1,1^-))$. Now, we are dealing with the Frobenius matrix $F(\varphi)$ for an F-crystal *non-singular at* 0 *and* 1, so we can forget the restriction that φ be adapted to the singularities, that is we can replace φ in the equality with the standard Frobenius $\lambda\mapsto\lambda^p$. Evaluation at $\lambda=1$ then gives

$$\xi_1^{(0)}(\vec{a},\vec{a}')\frac{a_3}{a_3'}\mathcal{F}(\vec{a};1)=\xi_1^{(1)}(\vec{a},\vec{a}')\frac{a_1-a_3}{a_1'-a_3'}$$

i.e.

$$\mathcal{F}(\vec{a};1)=\frac{a_3'}{a_3}\cdot\frac{a_1-a_3}{a_1'-a_3'}\cdot\frac{\xi_1^{(1)}(\vec{a},\vec{a}')}{\xi_1^{(0)}(\vec{a},\vec{a}')}\,.$$

This equality is essentially the Koblitz-Diamond formula. In fact, for $p\vec{b}-\vec{a}=\vec{\mu}$, we can rewrite the r.h.s. as

$$\begin{aligned}
&\frac{b_3}{a_3}\cdot\frac{a_1-a_3}{b_1-b_3}\cdot\frac{\xi_1^{(1)}(\vec{a},\vec{b})}{\xi_1^{(0)}(\vec{a},\vec{b})}=\\
&=(-)^{\mu_2}\frac{b_3}{a_3}\cdot\frac{a_1-a_3}{b_1-b_3}\cdot\frac{\xi_1^{(0)}(a_1,a_2,a_1+a_2-a_3,b_1,b_2,b_1+b_2-b_3)}{\xi_1^{(0)}(\vec{a},\vec{b})}\\
&=(-)^{\mu_2}\frac{b_3}{a_3}\cdot\frac{a_1-a_3}{b_1-b_3}\cdot\frac{\gamma_p(a_2,b_2)\gamma_p(a_1-a_3,b_1-b_3)\gamma_p(1+a_3,1+b_3)}{\gamma_p(a_1+a_2-a_3+1,b_1+b_2-b_3+1)\gamma_p(a_2,b_2)\gamma_p(a_3-a_2,b_3-b_2)}\\
&=(-)^{\mu_2}\frac{\gamma_p(a_1-a_3+1,b_1-b_3+1)\gamma_p(a_3,b_3)}{\gamma_p(a_1+a_2-a_3+1,b_1+b_2-b_3+1)\gamma_p(a_3-a_2,b_3-b_2)}\\
&=(-)^{\mu_2}\frac{\pi^{\mu_1-\mu_3+1}\Gamma_p(a_1-a_3+1)\pi^{\mu_3}\Gamma_p(a_3)}{\pi^{\mu_1+\mu_2-\mu_3+1}\Gamma_p(a_1+a_2-a_3+1)\pi^{\mu_3-\mu_2}\Gamma_p(a_3-a_2)}\\
&=(-)^{\mu_2}\frac{\Gamma_p(a_1-a_3+1)\Gamma_p(a_3)}{\Gamma_p(a_1+a_2-a_3+1)\Gamma_p(a_3-a_2)}\\
&=(-)^{\mu_2}\frac{(-)^{\mu_1-\mu_3}}{(-)^{\mu_1+\mu_2-\mu_3}}\frac{\Gamma_p(a_3-a_2-a_1)\Gamma_p(a_3)}{\Gamma_p(a_3-a_1)\Gamma_p(a_3-a_2)}\\
&=\frac{\Gamma_p(a_3-a_2-a_1)\Gamma_p(a_3)}{\Gamma_p(a_3-a_1)\Gamma_p(a_3-a_2)}\,.
\end{aligned}$$

4. Appendix: determination of Frobenius eigenvalues *via* modular relations.

4.1. Our calculations are based on the Dwork's article [Bo] §3. In particular, we recall the interplay of the base change matrices $B(\vec{a},\vec{b};\lambda)$, for $\vec{a}\equiv\vec{b}\bmod\mathbb{Z}$, of *loc.cit.*, with solutions

$$C_{\vec{a}}(z,\lambda)B(\vec{a},\vec{a}+\vec{u};\lambda)^t=B(\vec{a},\vec{a}+\vec{u};z)^tC_{\vec{a}+\vec{u}}(z,\lambda)$$

and with Frobenius matrices

$$B(\vec{a},\vec{a}+\vec{u};\lambda)\gamma^{(\varphi)}(\vec{a},\vec{b};\lambda) = \gamma^{(\varphi)}(\vec{a}+\vec{u},\vec{b}+\vec{v};\lambda)B(\vec{b},\vec{b}+\vec{v};\varphi(\lambda))$$

(the second formula follows from the first using the Frobenius action **3.0.1** on solutions).

4.1.1. For $z = 0, 1, \infty$, we use the solution matrix of **3.0.2**

$$C_{\vec{a}}(z,\lambda) = l_z(\lambda)^{D_z(\vec{a})}U_{\vec{a}}^{(z)}(\lambda)$$

where $l_z(\lambda) = \lambda$, $1-\lambda$, λ^{-1}, and

$$D_z(\vec{a}) = \begin{pmatrix} 0 & 0 \\ 0 & -a_3 \end{pmatrix} , \quad \begin{pmatrix} 0 & 0 \\ 0 & a_3-a_1-a_2 \end{pmatrix} , \quad \begin{pmatrix} a_2 & \\ 0 & a_3 \end{pmatrix} ,$$

respectively. We notice that $B(\vec{a},\vec{a}+\vec{u};z)^t C_{\vec{a}+\vec{u}}(z,\lambda)$ is a solution at z of the system $\frac{dY}{d\lambda} = YG_{\vec{a}+\vec{u}}(\lambda)$, so that we can write

$$l_z(\lambda)^{D_z(\vec{a})}U_{\vec{a}}^{(z)}(\lambda)B(\vec{a},\vec{a}+\vec{u};\lambda)^t = \Delta l_z(\lambda)^{D_z(\vec{a}+\vec{u})}U_{\vec{a}+\vec{u}}^{(z)}(\lambda)$$

with Δ diagonal and, using that $l_z(\lambda)^{D_z(\vec{a})}$ is a diagonal matrix, the formula gives

$$U_{\vec{a}+\vec{u}}^{(z)}(\lambda) = \begin{pmatrix} \alpha_1^{(z)}(\vec{a},\vec{u}) & 0 \\ 0 & \alpha_2^{(z)}(\vec{a},\vec{u}) \end{pmatrix} l_z(\lambda)^{-D_z(\vec{u})}U_{\vec{a}}^{(z)}(\lambda)B(\vec{a},\vec{a}+\vec{u};\lambda)^t \;. \tag{**4.1.2**}$$

4.2 From this we make explicit calculations at the singular points using the solutions written in **3.0.2**.

4.2.1. Lemma. *With the previous notation we have the following values for the terms $\alpha_i^{(z)}$: at the origin*

$$\alpha_1^{(0)}(\vec{a},\vec{e}_1) = 1 \qquad \alpha_2^{(0)}(\vec{a},\vec{e}_1) = \frac{a_1}{a_1-a_3}$$
$$\alpha_1^{(0)}(\vec{a},\vec{e}_2) = \frac{a_3-a_2-1}{a_2} \qquad \alpha_2^{(0)}(\vec{a},\vec{e}_2) = -1$$
$$\alpha_1^{(0)}(\vec{a},\vec{e}_3) = \frac{a_3+1}{a_3-a_2} \qquad \alpha_2^{(0)}(\vec{a},\vec{e}_3) = \frac{a_1-a_3-1}{a_3-1}$$

at 1

$$\alpha_1^{(1)}(\vec{a},\vec{e}_1) = \frac{a_1+a_2-a_3+1}{a_1-a_3} \qquad \alpha_2^{(1)}(\vec{a},\vec{e}_1) = \frac{a_1}{a_1+a_2-a_3-1}$$
$$\alpha_1^{(1)}(\vec{a},\vec{e}_2) = \frac{a_3-a_1-a_2-1}{a_2} \qquad \alpha_2^{(1)}(\vec{a},\vec{e}_2) = \frac{a_2-a_3+1}{a_3-a_1-a_2+1}$$
$$\alpha_1^{(1)}(\vec{a},\vec{e}_3) = \frac{a_1-a_3-1}{a_1+a_2-a_3} \qquad \alpha_2^{(1)}(\vec{a},\vec{e}_3) = \frac{a_1+a_2-a_3-2}{a_3-a_2}$$

and at infinity

$$\alpha_1^{(\infty)}(\vec{a},\vec{e}_1) = \frac{a_1-a_2+1}{a_3-a_1} \qquad \alpha_2^{(\infty)}(\vec{a},\vec{e}_1) = \frac{a_1}{a_1-a_2-1}$$
$$\alpha_1^{(\infty)}(\vec{a},\vec{e}_2) = \frac{a_3-a_2-1}{a_2-a_1} \qquad \alpha_2^{(\infty)}(\vec{a},\vec{e}_2) = \frac{a_2-a_1+2}{a_2}$$
$$\alpha_1^{(\infty)}(\vec{a},\vec{e}_3) = \frac{a_3-a_1+1}{a_3-a_2} \qquad \alpha_2^{(\infty)}(\vec{a},\vec{e}_3) = 1 \;.$$

Proof. The computation follows in every case the strategy of the previous section. For example, in the calculation of $\alpha_i^{(0)}(\vec{a},\vec{e}_1)$, we specialize $\vec{u} = \vec{e}_1 = (1,0,0)$ so $\vec{a}+\vec{u} = (a_1+1, a_2, a_3)$. From [Bo] we read

$$B(\vec{a},\vec{a}+\vec{e}_1;\lambda)^t = \frac{1}{a_1}\begin{pmatrix} a_1-a_3 & (a_1-a_3)\frac{\lambda}{\lambda-1} \\ a_3-a_2 & \frac{a_1-\lambda(a_3-a_2)}{1-\lambda} \end{pmatrix}$$

and we can specialize our general formula (**4.1.2**) to

$$U^{(0)}_{\vec{a}+\vec{e}_1}(\lambda) = \begin{pmatrix} \alpha^{(0)}_1(\vec{a},\vec{e}_1) & 0 \\ 0 & \alpha^{(0)}_2(\vec{a},\vec{e}_1) \end{pmatrix} \mathbb{I}_2 U^{(0)}_{\vec{a}}(\lambda) B(\vec{a},\vec{a}+\vec{e}_1;\lambda)^t$$

(the singular parts in this case disappear). Using the solution matrix of **3.0.2** evaluated for $\lambda = 0$ (*i.e.* we consider only the constant term of the expansion w.r.t. λ), we obtain:

$$\begin{pmatrix} a_3 - a_2 & a_3 \\ 1 - a_3 & 0 \end{pmatrix} = \begin{pmatrix} \alpha^{(0)}_1(\vec{a},\vec{e}_1) & 0 \\ 0 & \alpha^{(0)}_2(\vec{a},\vec{e}_1) \end{pmatrix} \begin{pmatrix} a_3 - a_2 & a_3 \\ 1 - a_3 & 0 \end{pmatrix} \frac{1}{a_1} \begin{pmatrix} a_1 - a_3 & 0 \\ a_3 - a_2 & a_1 \end{pmatrix}$$

from which the results for $\alpha^{(0)}_1(\vec{a},\vec{e}_1)$ and $\alpha^{(0)}_2(\vec{a},\vec{e}_1)$ follow:

$$a_1 \begin{pmatrix} a_3 - a_2 & a_3 \\ 1 - a_3 & 0 \end{pmatrix} = \begin{pmatrix} \alpha^{(0)}_1(\vec{a},\vec{e}_1) & 0 \\ 0 & \alpha^{(0)}_2(\vec{a},\vec{e}_1) \end{pmatrix} \begin{pmatrix} a_1(a_3 - a_2) & a_1 a_3 \\ (1 - a_3)(a_1 - a_3) & 0 \end{pmatrix} .$$

In other cases a more precise evaluation of the matrix solution is necessary. For example, in the calculation of $\alpha^{(1)}_i(\vec{a},\vec{e}_1)$ we can specialize our general formula (**4.1.2**) to

$$U^{(1)}_{\vec{a}+\vec{e}_1}(\lambda) = \begin{pmatrix} \alpha^{(1)}_1(\vec{a},\vec{e}_1) & 0 \\ 0 & \alpha^{(1)}_2(\vec{a},\vec{e}_1) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (1-\lambda) \end{pmatrix} U^{(1)}_{\vec{a}}(\lambda) B(\vec{a},\vec{a}+\vec{e}_1;\lambda)^t .$$

In this case we use the evaluation at $\lambda = 1$ and it is convenient to write the matrix $B(\vec{a},\vec{a}+\vec{e}_1;\lambda)^t$ in the form

$$B(\vec{a},\vec{a}+\vec{e}_1;\lambda)^t = \frac{1}{a_1} \begin{pmatrix} a_1 - a_3 & (a_1 - a_3) + (a_1 - a_3)\frac{1}{\lambda - 1} \\ a_3 - a_2 & (a_3 - a_2) + (a_1 + a_2 - a_3)\frac{1}{1-\lambda} \end{pmatrix} .$$

We need the lowest order term in the expansion w.r.t. $(1-\lambda)$ for the solution given in **3.0.2**. We obtain the equality

$$a_1 \begin{pmatrix} a_1 + a_2 - a_3 + 1 & a_1 - a_3 + 1 \\ 0 & a_1 + a_2 - a_3 \end{pmatrix} =$$
$$= \begin{pmatrix} \alpha^{(1)}_1(\vec{a},\vec{e}_1) & 0 \\ 0 & \alpha^{(1)}_2(\vec{a},\vec{e}_1) \end{pmatrix} \begin{pmatrix} a_1(a_1 - a_3) & a_1(a_1 - a_3)\frac{a_1 - a_3 + 1}{a_1 + a_2 - a_3 + 1} \\ 0 & (a_1 + a_2 - a_3)(a_1 + a_2 - a_3 + 1) \end{pmatrix}$$

from which the result follows. The other cases are obtained in a similar way. □

4.2.2. We have the obvious relations

$$\alpha^{(z)}_i(\vec{a},\vec{u}+\vec{v}) = \alpha^{(z)}_i(\vec{a}+\vec{u},\vec{v})\alpha^{(z)}_i(\vec{a},\vec{u}) \qquad \text{and} \qquad \alpha^{(z)}_i(\vec{a},\vec{0}) = 1 .$$

So, if we write

$$\alpha^{(z)}_i(\vec{a},\vec{u}) = \frac{\varphi^{(z)}_i(\vec{a}+\vec{u})}{\varphi^{(z)}_i(\vec{a})}$$

we have the following

LEMMA.

$$\varphi^{(0)}_1(\vec{a}) = \frac{\Gamma(a_3+1)}{\Gamma(a_2)\Gamma(a_3-a_2)} \qquad \varphi^{(0)}_2(\vec{a}) = (-)^{a_2}\frac{\Gamma(a_1)}{\Gamma(a_1-a_3)\Gamma(a_3-1)}$$

$$\varphi^{(1)}_1(\vec{a}) = (-)^{a_2}\frac{\Gamma(a_1+a_2-a_3+1)}{\Gamma(a_2)\Gamma(a_1-a_3)} \qquad \varphi^{(1)}_2(\vec{a}) = \frac{\Gamma(a_1)}{\Gamma(a_1+a_2-a_3-1)\Gamma(a_3-a_2)}$$

$$\varphi^{(\infty)}_1(\vec{a}) = (-)^{a_1+a_2+a_3}\frac{\Gamma(a_1-a_2+1)}{\Gamma(a_3-a_2)\Gamma(a_1-a_3)} \qquad \varphi^{(\infty)}_2(\vec{a}) = (-)^{a_2}\frac{\Gamma(a_1)}{\Gamma(a_1-a_2-1)\Gamma(a_2)} .$$

PROOF. This follows directly from the previous lemma, using the functional equation $\Gamma(x+1) = x\Gamma(x)$ (*i.e.* $\Gamma(x-1) = \Gamma(x)/(x-1)$) for the (classical) Gamma function. $\square$

4.3 We now deduce the modular properties of the Frobenius eigenvalues $\xi_i^{(z)}(\vec{a},\vec{b})$ in terms of the $\alpha_i^{(z)}(\vec{a},\vec{u})$. The Frobenius action of **3.0.1** adapted to a singular point, on solutions at the same point, gives

$$l_z(\varphi(\lambda))^{D_z(\vec{b})} U_{\vec{b}}^{(z)}(\varphi(\lambda))\gamma^{(\varphi)}(\vec{a},\vec{b};\lambda)^t = \begin{pmatrix} \xi_1^{(z)}(\vec{a},\vec{b}) & 0 \\ 0 & \xi_2^{(z)}(\vec{a},\vec{b}) \end{pmatrix} l_z(\lambda)^{D_z(\vec{a})} U_{\vec{a}}^{(z)}(\lambda)$$

and, the singular part of solutions being diagonal,

$$U_{\vec{b}}^{(z)}(\varphi(\lambda))\gamma^{(\varphi)}(\vec{a},\vec{b};\lambda)^t = \begin{pmatrix} \xi_1^{(z)}(\vec{a},\vec{b}) & 0 \\ 0 & \xi_2^{(z)}(\vec{a},\vec{b}) \end{pmatrix} l_z(\lambda)^{D_z(\vec{a})} l_z(\varphi(\lambda))^{-D_z(\vec{b})} U_{\vec{a}}^{(z)}(\lambda) .$$

We make the substitutions $\vec{a}$ to $\vec{a}+\vec{u}$ and $\vec{b}$ to $\vec{b}+\vec{v}$:

$$\begin{aligned} & U_{\vec{b}+\vec{v}}^{(z)}(\varphi(\lambda))\gamma^{(\varphi)}(\vec{a}+\vec{u},\vec{b}+\vec{v};\lambda)^t = \\ & \quad \begin{pmatrix} \xi_1^{(z)}(\vec{a}+\vec{u},\vec{b}+\vec{v}) & 0 \\ 0 & \xi_2^{(z)}(\vec{a}+\vec{u},\vec{b}+\vec{v}) \end{pmatrix} l_z(\lambda)^{D_z(\vec{a}+\vec{u})} l_z(\varphi(\lambda))^{-D_z(\vec{b}+\vec{v})} U_{\vec{a}+\vec{u}}^{(z)}(\lambda) , \end{aligned}$$

then we apply the translation formulas on solutions and Frobenius:

$$\begin{aligned} & \begin{pmatrix} \alpha_1^{(z)}(\vec{b},\vec{v}) & 0 \\ 0 & \alpha_2^{(z)}(\vec{b},\vec{v}) \end{pmatrix} l_z(\varphi(\lambda))^{D_z(\vec{b})-D_z(\vec{b}+\vec{v})} U_{\vec{b}}^{(z)}(\varphi(\lambda)) B(\vec{b},\vec{b}+\vec{v};\varphi(\lambda))^t \\ & \quad B(\vec{b},\vec{b}+\vec{v};\varphi(\lambda))^* \gamma^{(\varphi)}(\vec{a},\vec{b};\lambda)^t B(\vec{a},\vec{a}+\vec{u};\lambda)^t = \\ & = \begin{pmatrix} \xi_1^{(z)}(\vec{a}+\vec{u},\vec{b}+\vec{v}) & 0 \\ 0 & \xi_2^{(z)}(\vec{a}+\vec{u},\vec{b}+\vec{v}) \end{pmatrix} l_z(\lambda)^{D_z(\vec{a}+\vec{u})} l_z(\varphi(\lambda))^{-D_z(\vec{b}+\vec{v})} \\ & \quad \begin{pmatrix} \alpha_1^{(z)}(\vec{a},\vec{u}) & 0 \\ 0 & \alpha_2^{(z)}(\vec{a},\vec{u}) \end{pmatrix} l_z(\lambda)^{D_z(\vec{a})-D_z(\vec{a}+\vec{u})} U_{\vec{a}}^{(z)}(\lambda) B(\vec{a},\vec{a}+\vec{u};\lambda)^t . \end{aligned}$$

We finally make the possible simplifications and use Frobenius on the left

$$\begin{aligned} & \begin{pmatrix} \alpha_1^{(z)}(\vec{b},\vec{v}) & 0 \\ 0 & \alpha_2^{(z)}(\vec{b},\vec{v}) \end{pmatrix} \begin{pmatrix} \xi_1^{(z)}(\vec{a},\vec{b}) & 0 \\ 0 & \xi_2^{(z)}(\vec{a},\vec{b}) \end{pmatrix} l_z(\lambda)^{D_z(\vec{a})} U_{\vec{a}}^{(z)}(\lambda) = \\ & = \begin{pmatrix} \xi_1^{(z)}(\vec{a}+\vec{u},\vec{b}+\vec{v}) & 0 \\ 0 & \xi_2^{(z)}(\vec{a}+\vec{u},\vec{b}+\vec{v}) \end{pmatrix} \begin{pmatrix} \alpha_1^{(z)}(\vec{a},\vec{u}) & 0 \\ 0 & \alpha_2^{(z)}(\vec{a},\vec{u}) \end{pmatrix} l_z(\lambda)^{D_z(\vec{a})} U_{\vec{a}}^{(z)}(\lambda) . \end{aligned}$$

From this formula we deduce

$$\begin{aligned} & \begin{pmatrix} \alpha_1^{(z)}(\vec{b},\vec{v}) & 0 \\ 0 & \alpha_2^{(z)}(\vec{b},\vec{v}) \end{pmatrix} \begin{pmatrix} \xi_1^{(z)}(\vec{a},\vec{b}) & 0 \\ 0 & \xi_2^{(z)}(\vec{a},\vec{b}) \end{pmatrix} = \\ & = \begin{pmatrix} \xi_1^{(z)}(\vec{a}+\vec{u},\vec{b}+\vec{v}) & 0 \\ 0 & \xi_2^{(z)}(\vec{a}+\vec{u},\vec{b}+\vec{v}) \end{pmatrix} \begin{pmatrix} \alpha_1^{(z)}(\vec{a},\vec{u}) & 0 \\ 0 & \alpha_2^{(z)}(\vec{a},\vec{u}) \end{pmatrix} \end{aligned}$$

that is

$$\xi_i^{(z)}(\vec{a}+\vec{u},\vec{b}+\vec{v}) = \frac{\alpha_i^{(z)}(\vec{b},\vec{v})}{\alpha_i^{(z)}(\vec{a},\vec{u})} \xi_i^{(z)}(\vec{a},\vec{b})$$

for $i = 1, 2$, which express the modular properties of $\xi_i^{(z)}(\vec{a},\vec{b})$.

4.3.1. Lemma. *Let $\vartheta(\vec a,\vec b)$ be a function defined for all $(\vec a,\vec b)\in D(0,1)^6$, with $p\vec b-\vec a\in\mathbb{Z}^3$. We assume that ϑ is a p-adic analytic function of $(\vec a,\vec b)$, for fixed $\vec\mu=p\vec b-\vec a\in\mathbb{Z}^3$, subject to the modular properties*

$$\vartheta(\vec a+\vec u,\vec b+\vec v)=\frac{\alpha(\vec b,\vec v)}{\alpha(\vec a,\vec u)}\vartheta(\vec a,\vec b)$$

for every $u,v\in\mathbb{Z}^3$ with

$$\alpha(\vec a,\vec u)=\frac{\varphi(\vec a+\vec u)}{\varphi(\vec a)}$$

and

$$\varphi(\vec a)=\frac{\Gamma(\ell_1(\vec a))}{\Gamma(\ell_2(\vec a))\Gamma(\ell_3(\vec a))}$$

where the $\ell_i(\vec a)$ are linear functions of a_1,a_2,a_3. Then, up to a multiplicative constant, ϑ is of the form

$$\vartheta(\vec a,\vec b)=\frac{\gamma_p(\ell_2(\vec a),\ell_2(\vec b))\gamma_p(\ell_3(\vec a),\ell_3(\vec b))}{\gamma_p(\ell_1(\vec a),\ell_1(\vec b))}\ .$$

Proof. It is sufficient to check the modular properties for $\vartheta(\vec a,\vec b)$: we have

$$\frac{\vartheta(\vec a+\vec u,\vec b+\vec v)}{\vartheta(\vec a,\vec b)}=\frac{\varphi(\vec a)}{\varphi(\vec a+\vec u)}\frac{\varphi(\vec b+\vec v)}{\varphi(\vec b)}=$$
$$=\frac{\Gamma(\ell_1(\vec a))}{\Gamma(\ell_2(\vec a))\Gamma(\ell_3(\vec a))}\frac{\Gamma(\ell_2(\vec a+\vec u))\Gamma(\ell_3(\vec a+\vec u))}{\Gamma(\ell_1(\vec a+\vec u))}\frac{\Gamma(\ell_1(\vec b+\vec v))}{\Gamma(\ell_2(\vec b+\vec v))\Gamma(\ell_3(\vec b+\vec v))}\frac{\Gamma(\ell_2(\vec b))\Gamma(\ell_3(\vec b))}{\Gamma(\ell_1(\vec b))}$$
$$=\frac{\Gamma(\ell_1(\vec a))\Gamma(\ell_1(\vec b)+\ell_1(\vec v))}{\Gamma(\ell_1(\vec a)+\ell_1(\vec u))\Gamma(\ell_1(\vec b))}\frac{\Gamma(\ell_2(\vec a)+\ell_2(\vec u))\Gamma(\ell_2(\vec b))}{\Gamma(\ell_2(\vec a))\Gamma(\ell_2(\vec b)+\ell_2(\vec v))}\frac{\Gamma(\ell_3(\vec a)+\ell_3(\vec u))\Gamma(\ell_3(\vec b))}{\Gamma(\ell_3(\vec a))\Gamma(\ell_3(\vec b)+\ell_3(\vec v))}$$

and the result follows using the modular properties of the function γ_p. □

4.3.2. Corollary. *For $\vec b\in D(0,1)^3$, $p\vec b-\vec a\in\mathbb{Z}^3$, then:*

$$\xi_1^{(0)}(\vec a,\vec b)=\frac{\gamma_p(a_2,b_2)\gamma_p(a_3-a_2,b_3-b_2)}{\gamma_p(a_3+1,b_3+1)}$$
$$\xi_2^{(0)}(\vec a,\vec b)=(-)^{\mu_2}\frac{\gamma_p(a_1-a_3,b_1-b_3)\gamma_p(a_3-1,b_3-1)}{\gamma_p(a_1,b_1)}$$
$$\xi_1^{(1)}(\vec a,\vec b)=(-)^{\mu_2}\frac{\gamma_p(a_2,b_2)\gamma_p(a_1-a_3,b_1-b_3)}{\gamma_p(a_1+a_2-a_3+1,b_1+b_2-b_3+1)}$$
$$\xi_2^{(1)}(\vec a,\vec b)=\frac{\gamma_p(a_1+a_2-a_3-1,b_1+b_2-b_3-1)\gamma_p(a_3-a_2,b_3-b_2)}{\gamma_p(a_1,b_1)}$$
$$\xi_1^{(\infty)}(\vec a,\vec b)=(-)^{\mu_1+\mu_2+\mu_3}\frac{\gamma_p(a_3-a_2,b_3-b_2)\gamma_p(a_1-a_3,b_1-b_3)}{\gamma_p(a_1-a_2+1,b_1-b_2+1)}$$
$$\xi_2^{(\infty)}(\vec a,\vec b)=(-)^{\mu_2}\frac{\gamma_p(a_1-a_2-1,b_1-b_2-1)\gamma_p(a_2,b_2)}{\gamma_p(a_1,b_1)}$$

up to multiplicative constants.

Proof. Use the previous lemma and the modular properties of the functions $\xi_i^{(z)}$ for $i=1,2$ and $z\in\{0,1,\infty\}$. □

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